TSKS14 Multiple Antenna Communications

Solutions for the exam 2020-08-18

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1

a. The downlink SNR is computed as

$$\frac{P_{\mathrm{dl}}|g|^2}{BN_0} = \frac{10 \cdot 10^{-11}}{10^7 \cdot 10^{-17}} = 10^0 = 0\,\mathrm{dB}.$$

The uplink SNR is computed as

$$\frac{P_{\mathrm{ul}}|g|^2}{BN_0} = \frac{10^{-1} \cdot 10^{-11}}{10^7 \cdot 10^{-17}} = 10^{-2} = -20\,\mathrm{dB}.$$

b. The uplink SNR is -20 dB which is below the required -10 dB. We need to increase the SNR by at least 10 dB.

(a) By using multiple antennas at the base station and maximum ratio combining, the uplink SNR will grow proportionally with the number of antennas M. With perfect channel knowledge, we need $M = 10$ antennas to increase the SNR by 10 dB. In practice, more antennas are needed to compensate for imperfect channel knowledge and variations in the channel gain between different antennas. One can also use more than 10 antennas to increase the SNR by more than 10 dB.

A good consequence of using multiple antennas at the base station is that also the downlink SNR increases with the same number, so the capacity will increase in both directions.

(b) The uplink SNR is inversely proportional to the bandwidth. Hence, if we decrease the bandwidth but keep the transmit power fixed, we can increase the SNR since the power spectral density of the signal (W/Hz) increases. If we only use 1 MHz of bandwidth in the uplink, we will get an SNR of -10 dB. The negative consequence of this is that the capacity is proportional to the bandwidth, but if the system is non-operational for transmit powers below -10 dB, there is no other choice. It is possible to use the full bandwidth in the downlink but only a part of the bandwidth in the uplink.

Answer: a) Uplink SNR = -20 dB, downlink SNR = 0 dB b) We can use 10 antennas (or more) to increase the uplink SNR to -10 dB (or higher). We can decrease the bandwidth by 10 times (or more) to achieve the same effect.

2

When transmitting with multiple antennas over a line-of-sight channel, the beamwidth describes how focused the transmission is around the angular direction of the receiver. The transmission will consist of a main beam/lobe that is focused at the receiver and smaller sidelobes in other directions. The width of the main beam is the beamwidth and it can be measured in degrees or in radians. One can define the width in different ways: 1) the half-power beamwidth where the beamforming gain is between M and $M/2$; 2) the amplification beamwidth where the beamforming gain is between M and 1; and 3) the first-null beamwidth that is the full width of the main beam from one null to another null.

The beamwidth is associated with the beamforming gain. The smaller the beamwidth is, the more the transmit power is focused in the direction of the receiver, which leads to a larger beamforming gain. A small beamwidth is particularly useful in multi-user MIMO since the interference between two users will be low if their respective main beams are non-overlapping.

An example of how to compute the first-null beamwidth with a uniform linear array of M isotropic antennas is provided in Section 3.2.2 in "Introduction to Multiple Antenna Communications". It can be repeated in the solution, using your own words. When transmitting in the broadside direction, the beamwidth is approximately $4/M$ radians.

If we want a beamwidth of 15°, which corresponds to approximately 0.26 radians, we need $M \approx 4/0.26 \approx 15.4$ antennas. So at least 16 antennas are required.

Answer: See above.

3

a. The capacity of a MISO channel is

$$C = \log_2(1 + \rho\|\boldsymbol{g}\|^2) \quad \text{bit/s/Hz},$$

where $\|\boldsymbol{g}\|^2 = 4$ for the given channel. We know that $1 + \rho\|\boldsymbol{g}\|^2 = 41$, from which we can obtain the SNR $\rho = 10$.

b. We achieve the capacity with maximum ratio transmission, where the precoding vector is

$$\boldsymbol{a} = \frac{\boldsymbol{g}^*}{\|\boldsymbol{g}\|} = \frac{1}{2}\left[1 \ e^{-j\pi\sin(\theta)} \ \ e^{-j2\pi\sin(\theta)} \ \ \ e^{-j3\pi\sin(\theta)}\right]^T.$$

This precoding vector depends on θ since the precoding should create constructive interference of the four transmitted signals in the angular direction θ that leads to the receiver. However, the channel capacity independent of θ since it assumes that the optimal precoding is used.

c. Let $\boldsymbol{g}(\theta)$ denote the channel vector as a function of the angle θ. When using an arbitrary precoding vector $\boldsymbol{a}$, the achievable information rate is

$$\log_2(1 + \rho|\boldsymbol{g}(\theta_1)^T\boldsymbol{a}|^2)$$

where we used the true angle θ_1. Since the base station believes that $\theta = 0^\circ$, the precoding vector from Part **b** becomes $\boldsymbol{a} = \frac{1}{2}[1\ 1\ 1\ 1]^T$. Hence, the achievable information rate becomes

$$\log_2\left(1 + \frac{\rho}{2}\left|1 + e^{j\pi\sin(\theta_1)} + e^{j2\pi\sin(\theta_1)} + e^{j3\pi\sin(\theta_1)}\right|^2\right).$$

In this case, we have $\theta_1 = 60^\circ$ and therefore the information rate is approximately 1.2952. This is substantially smaller than the capacity $\log_2(41) \approx 5.4$ which is obtained when the beamforming is optimal.

d. In this case, $\boldsymbol{g}(\theta_1) = [1\ 1j\ \ -1\ \ -1j]^T$ which is orthogonal to $\boldsymbol{a} = \frac{1}{2}[1\ 1\ 1\ 1]^T$, which means that the signal is transmitted in a completely different direction where there happen to be a null. Hence, the information rate is zero.

Answer: See above

4

a. All the terminals transmit with full power, have the same channel quality to their serving base station, and the same channel quality to the interfering base station. Hence, it is logical that the uplink SINR will be the same.

By using the table of formulas, the uplink effective SINR of an arbitrary terminal becomes

$$\mathrm{SINR}^{\mathrm{mr,ul}} = \frac{M\rho_{\mathrm{ul}}\gamma_{\mathrm{same}}}{1+\rho_{\mathrm{ul}}K(1+\alpha)+M\rho_{\mathrm{ul}}\gamma_{\mathrm{other}}}$$

where

$$\gamma_{\mathrm{same}} = \frac{K\rho_{\mathrm{ul}}}{1+K\rho_{\mathrm{ul}}(1+\alpha)},$$

$$\gamma_{\mathrm{other}} = \frac{K\rho_{\mathrm{ul}}\alpha^2}{1+K\rho_{\mathrm{ul}}(1+\alpha)}.$$

The resulting lower bound on the capacity is $\log_2(1+\mathrm{SINR}^{\mathrm{mr,ul}})$.

b. The net sum spectral efficiency is $(1-K/\tau_c)K\log_2(1+\mathrm{SINR}^{\mathrm{mr,ul}})$.

c. It follows directly that $\mathrm{SINR}^{\mathrm{mr,ul}} \to \frac{\gamma_{\mathrm{same}}}{\gamma_{\mathrm{other}}}$ as $M \to \infty$. Hence, the asymptotic limit is

$$\left(1-\frac{K}{\tau_c}\right)K\log_2\left(1+\frac{\gamma_{\mathrm{same}}}{\gamma_{\mathrm{other}}}\right) = \left(1-\frac{K}{\tau_c}\right)K\log_2\left(1+\frac{1}{\alpha^2}\right)$$

d. The number of users appear in the "prelog factor" $(1-K/\tau_c)K$. The first derivative of it is $1-2K/\tau_c$ which is zero for $K=\tau_c/2$. This is the maximum since the second derivative is negative. Hence, the optimum is to let the number of users per cell equal half the coherence interval. This also implies that half the coherence interval is spent on sending pilots and the other half on data transmission.

Answer: See above.

5

a. The outage probability is defined as

$$p_{out}(R) = \mathbb{P}\left\{\log_2(1+|g|^2P) < R\right\}$$

using the notation from the problem formulation. Since g is uniformly distributed between -1 and 1, it holds that $|g|$ is uniformly distributed between 0 and 1. Hence, we have that

$$\mathbb{P}\left\{|g|^2 < x\right\} = \mathbb{P}\left\{|g| < \sqrt{x}\right\} = \begin{cases} 0 & x < 0 \\ \sqrt{x} & x \in [0,1] \\ 1 & x > 1. \end{cases}$$

By utilizing this result, we can get

$$p_{out}(R) = \mathbb{P}\left\{|g|^2 < \frac{2^R-1}{P}\right\} = \begin{cases} \sqrt{\frac{2^R-1}{P}} & R \in [0, \log_2(1+P)] \\ 1 & R > \log_2(1+P) \end{cases}.$$

Note that we omitted the case $R<0$ since $R \geq 0$ according to the problem formulation (and since only positive rates make any sense).

b. With two antennas and maximum ratio combining, which is capacity achieving, the outage probability becomes

$$p_{\text{out}}(R) = \mathbb{P}\left\{\log_2(1 + \|\mathbf{g}\|^2 P) < R\right\}$$

where $\mathbf{g} = [g\, g]^T$ and thus $\|\mathbf{g}\|^2 = 2|g|^2$.

Following the same approach as in Part **a**, we get

$$p_{out}(R) = \mathbb{P}\left\{|g|^2 < \frac{2^R - 1}{2P}\right\} = \begin{cases} \sqrt{\frac{2^R-1}{2P}} & R \in [0, \log_2(1+2P)] \\ 1 & R > \log_2(1+2P) \end{cases}.$$

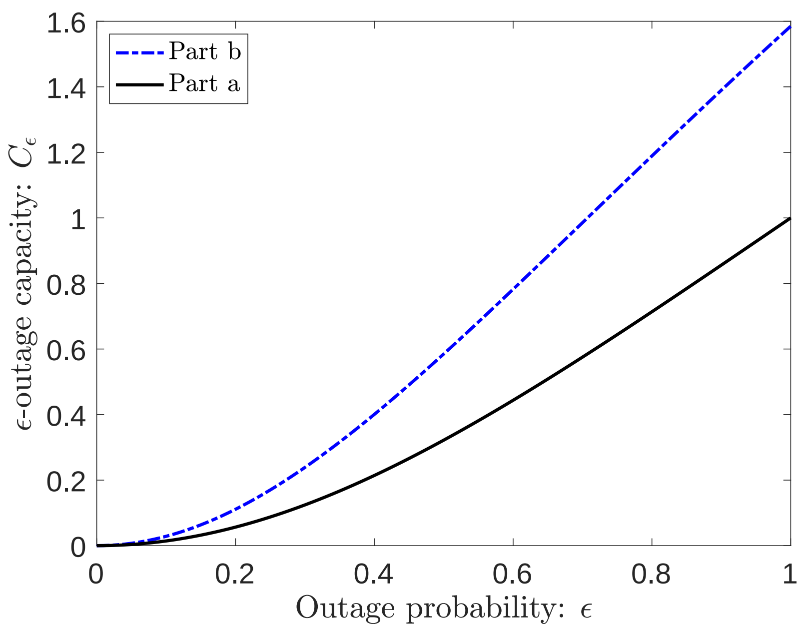
c. In Part **a**, the ϵ-outage capacity is obtained by setting $\epsilon = p_{out}(R) = \sqrt{\frac{2^R-1}{P}}$ and solving for R. We obtain

$$C_{\epsilon,(a)} = \log_2(1 + P\epsilon^2).$$

In Part **b**, we have obtained a beamforming gain and the same computations lead to

$$C_{\epsilon,(b)} = \log_2(1 + 2P\epsilon^2).$$

To sketch a graph, we need to select a value of P. Here is a graph for $P = 1$:



Answer: See above.