TSKS14 Multiple Antenna Communications

Solutions for the exam 2019-05-29

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1

a. This is a SIMO channel with perfect channel knowledge. Since each antenna receives the same fraction of the transmitted power, we have $\|\boldsymbol{g}\|^2 = M\frac{\lambda^2}{(4\pi)^2}\frac{1}{d^2}$.

The capacity is then given by

$$C = B\log_2\left(1 + \frac{PM}{BN_0}\frac{\lambda^2}{(4\pi)^2}\frac{1}{d^2}\right) \quad \text{bit/s}.$$

b. By treating the capacity expression in **a** as an equation, we can solve for d and obtain

$$d = \frac{\lambda}{4\pi}\sqrt{\frac{PM}{BN_0(2^{C/B}-1)}}.$$

By inserting the given numbers, we obtain $d \approx 795.77$ m.

c. By treating the capacity expression in **a** as an equation, we can solve for M and obtain

$$M = \left(\frac{4\pi d}{\lambda}\right)^2\frac{BN_0(2^{C/B}-1)}{P}.$$

By inserting the given numbers (in particular $C = 100$ Mbit/s and d from **b**), we obtain $M \approx 1023$.

d. The total antenna area is $MA_{\text{iso}} = M\frac{\lambda^2}{4\pi} \approx 0.81\ \text{m}^2$.

e. By using the same expression as in **c**, we now need $M = 102300$ antennas. However, $MA_{\text{iso}} = M\frac{\lambda^2}{4\pi} \approx 0.81\ \text{m}^2$ also in this case.

Answer: See above.

2

The answer should provide a reasonably correct description of the capacity analysis for slow fading channels. The following things can give one point each (other correct and important statements may also give points):

- Correct explanation of the slow and fast fading concepts (i.e., one random channel realization while transmitting under slow fading, infinitely many channel realizations under fast fading)
- Correct definition of the outage probability
- Correct definition of the outage capacity

- Example of how to compute the outage probability for channel of own choice.
- Correct explanation of how the use of multiple antennas improves the outage probability and/or outage capacity.

Answer: See above.

3

a. This is the capacity of a MISO channel, which is given by

$$C = \log_2(1 + \rho \|\boldsymbol{g}\|^2) = \log_2(1+8) \approx 3.17$$

since $\rho = 2$ and $\|\boldsymbol{g}\|^2 = M = 4$. This capacity expression does not depend on θ.

b. Let $\boldsymbol{g}(\theta)$ denote the channel vector as a function of the angle θ. When using an arbitrary precoding vector $\boldsymbol{a}$, the achievable information rate is

$$\log_2(1 + \rho |\boldsymbol{g}(\theta_1)^T \boldsymbol{a}|^2)$$

where we used the true angle θ_1.

Maximum ratio processing is given by $\boldsymbol{a} = \frac{\boldsymbol{g}^*(\theta)}{\|\boldsymbol{g}(\theta)\|}$. Since the base station believes that $\theta = 0°$, we get $\boldsymbol{a} = \frac{1}{\sqrt{M}}[1\ 1\ 1\ 1]^T$. Hence, the achievable information rate becomes

$$\log_2\left(1 + \frac{\rho}{M}\left|1 + e^{j\pi\sin(\theta_1)} + e^{j2\pi\sin(\theta_1)} + e^{j3\pi\sin(\theta_1)}\right|^2\right).$$

c. From the triangle inequality we know that

$$\left|1 + e^{j\pi\sin(\theta_1)} + e^{j2\pi\sin(\theta_1)} + e^{j3\pi\sin(\theta_1)}\right| \leq |1| + \left|e^{j\pi\sin(\theta_1)}\right| + \left|e^{j2\pi\sin(\theta_1)}\right| + \left|e^{j3\pi\sin(\theta_1)}\right| = M = 4$$

The upper bound is achieved only if all the terms in the sum are equal to one, which happens when $\sin(\theta_1) = 0$. Hence $\theta_1 = 0$ and $\theta_1 = \pi$ are the two angles that maximizes the information rate.

d. The smallest values is clearly equal to zero. Note that all the terms in the sum are of the form $e^{jm\pi\sin(\theta_1)}$ for $m = 0, 1, 2, 3$, which means that they are on the unit circle. We need these points to be equally spread out on the unit circle if they should cancel out. This happens if $\theta_1 = \pi/2$ and also when $\theta_1 = 3\pi/2$ (or equivalently $\theta_1 = -\pi/2$.

Answer: See above

4

The capacity lower bound that we are using is:

$$\log_2\left(1 + \frac{\rho_{\text{dl}}\eta_k |E\{\mathbf{g}_k^T \mathbf{a}_k\}|^2}{\sum_{k'=1}^{K} \rho_{\text{dl}}\eta_{k'} E\{|\mathbf{g}_k^T \mathbf{a}_{k'}|^2\} + 1 - \rho_{\text{dl}}\eta_k |E\{\mathbf{g}_k^T \mathbf{a}_k\}|^2}\right)$$

a. We need to compute all the expectations in the expression above. We start with $E\{\mathbf{g}_k^T \mathbf{a}_k\} = \sum_{m=1}^{M} E\{g_{km}\} = 0$ where g_{km} denotes the mth element in $\mathbf{g}_k$. Note that $E\{g_{km}\} = 0$ since the random variable is -1 and $+1$ with equal probability. This leads to the ratio inside the logarithm being zero.

We conclude that the capacity lower bound is zero in this case.

b. We need to compute all the expectations in the expression above. In this case, we get a non-zero expression since

$$E\{\mathbf{g}_k^T \mathbf{a}_k\} = E\{\|\mathbf{g}_k\|^2\} = \sum_{m=1}^{M} E\{|g_{km}|^2\} = M.$$

Note that $E\{|g_{km}|^2\} = 1$ since $|g_{km}| = 1$ both when the random variable is -1 and $+1$.

We also need to compute $E\{|\mathbf{g}_k^T \mathbf{a}_{k'}|^2\}$. We begin with the case of $k = k'$, which gives

$$E\{|\mathbf{g}_k^T \mathbf{a}_k|^2\} = E\{\|\mathbf{g}_k\|^4\} = E\left\{\left(\sum_{m=1}^{M} |g_{km}|^2\right)^2\right\} = M^2$$

since $|g_{km}|^2 = 1$.

In the case of $k \neq k'$, we have

$$E\{|\mathbf{g}_k^T \mathbf{a}_{k'}|^2\} = E\left\{\sum_{m=1}^{M}\sum_{n=1}^{M} g_{km} g_{kn} g_{k'm} g_{k'n}\right\} = E\left\{\sum_{m=1}^{M} g_{km} g_{km} g_{k'm} g_{k'm}\right\} = M$$

since only the case of $m = n$ gives a non-zero value of the expectation.

By substituting these values into the capacity lower bound, we obtain

$$\log_2\left(1 + \frac{\rho_{\mathrm{dl}} \eta_k M^2}{\sum\limits_{k'=1, k' \neq k}^{K} \rho_{\mathrm{dl}} \eta_{k'} M + 1}\right)$$

c. In this case, we know from the derivations in the course book that

$$E\{\mathbf{g}_k^T \mathbf{g}_k^*\} = M$$

$$E\{|\mathbf{g}_k^T \mathbf{g}_{k'}^*|^2\} = \begin{cases} M^2 + M & \text{if } k' = k \\ M & \text{if } k' \neq k. \end{cases}$$

By substituting these values into the capacity lower bound, we obtain

$$\log_2\left(1 + \frac{\rho_{\mathrm{dl}} \eta_k M^2}{\sum\limits_{k'=1}^{K} \rho_{\mathrm{dl}} \eta_{k'} M + 1}\right).$$

The only difference from the previous expression is that the desired user is included in the summation in the denominator. Hence, the case with i.i.d. CN(0,1) elements gives the lowest value of the capacity lower bound. The reason is the larger variation in the gain $\mathbf{g}_k^T \mathbf{a}_k$ when having Rayleigh fading.

Answer: See above.

5

a. The conventional expression from the table of formulas is

$$\mathrm{SINR}_{l1}^{\mathrm{mr,dl}} = \frac{M\rho_{\mathrm{dl}}\gamma_{l1}^{l}\eta_{l1}}{1+\rho_{\mathrm{dl}}\sum\limits_{l'=1}^{2}\beta_{l1}^{l'}\eta_{l'1} + M\rho_{\mathrm{dl}}\sum\limits_{l'=1,l'\neq l}^{2}\gamma_{l1}^{l'}\eta_{l'1}}$$

for the case of $K=1$ (and hence $k=1$ is the user of interest) and two cells that use the same pilot.

The M is the numerator is the number of antennas at base station l, since it is the beamforming gain of the signal transmission to the user in cell l. The M in denominator is the number of antennas at the base station $l' \neq l$, because it is the coherent interference caused by the other base station due to pilot contamination.

Hence, the SINR of the user in cell 1 is

$$\mathrm{SINR}_{11}^{\mathrm{mr,dl}} = \frac{M_1\rho_{\mathrm{dl}}\gamma_{11}^{1}\eta_{11}}{1+\rho_{\mathrm{dl}}\sum\limits_{l'=1}^{2}\beta_{11}^{l'}\eta_{l'1} + M_2\rho_{\mathrm{dl}}\gamma_{11}^{2}\eta_{21}}$$

and the SINR of the user in cell 2 is

$$\mathrm{SINR}_{21}^{\mathrm{mr,dl}} = \frac{M_2\rho_{\mathrm{dl}}\gamma_{21}^{2}\eta_{21}}{1+\rho_{\mathrm{dl}}\sum\limits_{l'=1}^{2}\beta_{21}^{l'}\eta_{l'1} + M_1\rho_{\mathrm{dl}}\gamma_{21}^{1}\eta_{11}}$$

b. By using the expressions from **a**, we observe that $\mathrm{SINR}_{11}^{\mathrm{mr,dl}} \to \infty$ and $\mathrm{SINR}_{21}^{\mathrm{mr,dl}} \to 0$. The reason is that the beamforming gain of the user in cell 1 grows with the number of antennas M_1 at its serving base station, while the pilot-contaminated coherent interference that is caused to the user in cell 2 grows with M_1 and makes the SINR go to zero.

c. By inserting $M_1 = c_1 M$ and $M_2 = c_2 M$ into the expressions and then taking the limit we obtain

$$\mathrm{SINR}_{11}^{\mathrm{mr,dl}} \to \frac{c_1\rho_{\mathrm{dl}}\gamma_{11}^{1}\eta_{11}}{c_2\rho_{\mathrm{dl}}\gamma_{11}^{2}\eta_{21}}$$

and

$$\mathrm{SINR}_{21}^{\mathrm{mr,dl}} = \frac{c_2\rho_{\mathrm{dl}}\gamma_{21}^{2}\eta_{21}}{c_1\rho_{\mathrm{dl}}\gamma_{21}^{1}\eta_{11}}.$$

Since the numbers of antennas at both base stations are growing, both the beamforming gain and the coherent interference from pilot contamination grows towards infinity. The consequence is that the non-coherent interference and noise vanish asymptotically.

d. We need to utilize the expressions for γ_{11}^{1}, γ_{11}^{2}, γ_{21}^{1}, and γ_{21}^{2} since they contain the uplink power and it decreases as $1/M$. In particular, $M\gamma_{11}^{1} \to (\beta_{11}^{1})^2$, $M\gamma_{11}^{2} \to (\beta_{11}^{2})^2$, $M\gamma_{21}^{1} \to (\beta_{21}^{1})^2$, and $M\gamma_{21}^{2} \to (\beta_{21}^{2})^2$. Hence, the limits are

$$\mathrm{SINR}_{11}^{\mathrm{mr,dl}} = \frac{c_1\rho_{\mathrm{dl}}(\beta_{11}^{1})^2\eta_{11}}{1+\rho_{\mathrm{dl}}\sum\limits_{l'=1}^{2}\beta_{11}^{l'}\eta_{l'1} + c_2\rho_{\mathrm{dl}}(\beta_{11}^{2})^2\eta_{21}}$$

and

$$\mathrm{SINR}_{21}^{\mathrm{mr,dl}} = \frac{c_2\rho_{\mathrm{dl}}(\beta_{21}^{2})^2\eta_{21}}{1+\rho_{\mathrm{dl}}\sum\limits_{l'=1}^{2}\beta_{21}^{l'}\eta_{l'1} + c_1\rho_{\mathrm{dl}}(\beta_{21}^{1})^2\eta_{11}}.$$

Answer: