TSKS14 Multiple Antenna Communications

Solutions for the exam 2018-11-01

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1

a. Following the derivation in Section 5.3.1 of "Fundamentals of Wireless Communications", the capacity is

$$C = \log_2(1 + P\|\boldsymbol{g}\|^2) = \log_2(1 + PM) \text{ bit/s/Hz}.$$

This capacity is achieved if $x[m] \sim CN(0, P)$.

b. You need $(2^6 - 1)/P = 12.6$ antennas. The smallest integer number is therefore $M = 13$.

c. You need a power of $P = (2^6 - 1)/50 = 1.26$.

d. The sign of the elements does not matter when computing $\|\boldsymbol{g}\|^2$. We thus have $\|\boldsymbol{g}\|^2 = M$ just as in Part **a**. The capacity is still $\log_2(1 + PM)$ bit/s/Hz.

Answer: See above.

2

A coherence interval is a part of the time-frequency space in which a channel can be considered constant and frequency-flat, so that the channel can be described by constant multiplication with a scalar/vector/matrix. The number of symbols in a coherence interval is $\tau_c = T_\mathrm{c} B_\mathrm{c}$, where T_c is the coherence time and B_c is the coherence bandwidth.

The coherence time is affected by how quickly things around the transmitter and the received move (or how quickly the transceivers move). The quicker they more, the faster the channel will change. A rule of thumb is that T_c is in the order of $\frac{\lambda}{2v}$, where λ is the wavelength of the carrier and v is the speed of fastest moving object in the vicinity of the transceivers.

The coherence bandwidth is affected by the time difference between the received signal components in a multipath environment. The larger the difference is between the shortest and the longest path (measured in time by the delay spread T_d), the smaller is the coherence bandwidth. A rule of thumb is that B_c is in the order of c/T_d, where c is the speed of light and T_d is the delay spread.

The coherence interval concept is important in massive MIMO because, in order to use the many antennas, the base station first estimates the channel. This channel estimate is only valid for τ_c samples (channel uses), and then the base station has to estimate the channel again. The finite coherence interval dimensionality is the reason why massive MIMO need TDD operation in order to handle a large number of antennas, since in TDD the number of pilots is proportional to the number of users. Pilots, uplink data, and downlink data must all fit into one coherence interval.

Answer: See above.

3

a. The capacity of a point-to-point MIMO channel is achieved by the waterfilling power allocation. From the course book, we know that the capacity is

$$C = \sum_{i=1}^{2} \log_2(1 + \rho q_i d_i)$$

where d_i is the ith eigenvalue of $\boldsymbol{G}^H\boldsymbol{G}$ and q_1, q_2 are obtained from the waterfilling algorithm.

In this case,

$$\boldsymbol{G}^H\boldsymbol{G} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

and thus $d_1 = d_2 = 2$. Since the eigenvalues are equal, there is no reason for one eigenvalue to get more transmit power than the other; the expressions are equal in the waterfilling algorithm. Hence, the power must be equally distributed: $q_1 = q_2 = 1/2$. The capacity then becomes

$$C = 2\log_2\left(1 + 2 \cdot \frac{1}{2} \cdot 2\right) = 2\log_2(3) \approx 3.17$$

b. We can use the same methodology, but now

$$\boldsymbol{G}^H\boldsymbol{G} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$$

and thus $d_1 = 2$ and $d_2 = 8$.

We know need to apply the waterfilling algorithm to find q_1, q_2. It says that we should find the largest $\mu \geq 0$ such that $q_1 + q_2 = 1$ and $q_i = (\mu - \frac{1}{\rho d_i})^+$.

If $0 < \mu < \frac{1}{4}$, we will get $q_1 = 0$, but in this case we have $q_2 < \frac{3}{16} < 1$ so the constraint $q_1 + q_2 = 1$ is not satisfied. Consequently, we need to have $\mu > \frac{1}{4}$ and then $q_1 > 0$ and $q_2 > 0$.

We then have $1 = q_1 + q_2 = \mu - \frac{1}{4} + \mu - \frac{1}{16} = 2\mu - \frac{5}{16}$, from which we get $\mu = \frac{21}{32}$ and eventually $q_1 = \frac{13}{32}$ and $q_2 = \frac{19}{32}$. The capacity then becomes

$$C = \log_2\left(1 + 2 \cdot \frac{13}{32} \cdot 2\right) + \log_2\left(1 + 2 \cdot \frac{19}{32} \cdot 8\right) \approx 4.78$$

c. The capacity is achieved by sending a Gaussian signal $\boldsymbol{x} = [x_1\, x_2]^T$, where $x_1 \sim \mathrm{CN}(0, 13/32)$ and $x_2 \sim \mathrm{CN}(0, 19/32)$ are independent (The SNR is excluded from these expressions.)

Answer: a) $C \approx 3.17$, b) $C \approx 4.78$, c) $\boldsymbol{x} = [x_1\ x_2]^T$, where $x_1 \sim \mathrm{CN}(0, 13/32)$ and $x_2 \sim \mathrm{CN}(0, 19/32)$.

4

a. Figure C.4 in "Fundamentals of Massive MIMO" is what should be drawn and one needs to point out which equation that gives each of the three sides (called (C.40)-(C.42) in that book).

b. For this to happen, we need

$$\log_2(1 + P\|\boldsymbol{g}_1\|^2) + \log_2(1 + P\|\boldsymbol{g}_2\|^2) = \log_2|\boldsymbol{I}_K + P\boldsymbol{G}^H\boldsymbol{G}|.$$

By expanding the determinant of the two-by-two matrix inside the last logarithm, we get

$$\log_2|\boldsymbol{I}_K + P\boldsymbol{G}^H\boldsymbol{G}| = \log_2\left((1 + P\|\boldsymbol{g}_1\|^2)(1 + P\|\boldsymbol{g}_2\|^2) - P^2|\boldsymbol{g}_2^H\boldsymbol{g}_1|^2\right)$$

where the largest value that the term $-P^2|\boldsymbol{g}_2^H\boldsymbol{g}_1|^2$ can take is zero, which is achieved when $\boldsymbol{g}_2^H\boldsymbol{g}_1 = 0$. In this case, the right hand side becomes $\log_2(1 + P\|\boldsymbol{g}_1\|^2) + \log_2(1 + P\|\boldsymbol{g}_2\|^2)$. Hence, $\boldsymbol{g}_2^H\boldsymbol{g}_1 = 0$ is the condition that we are looking for. It means that the two vectors are orthogonal and this condition has also been called favorable propagation in the course.

c. If M is even, then $\boldsymbol{g}_2^H \boldsymbol{g}_1 = 0$. In that case, the sum rate is $2\log_2(1+M)$ and we need $M \geq (2^{10/2} - 1) = 31$ to give the required performance. This is satisfied for $M = 32, 34, \ldots$.

If M is odd, then $\boldsymbol{g}_2^H \boldsymbol{g}_1 = 1$ and the sum rate $\log_2((1+M)^2 - 1)$ and we need $M \geq \sqrt{2^{10}+1} - 1 \approx 31.0156$. This is satisfied for $M = 33, 35, \ldots$.

In summary, the condition is satisfied for all $M \geq 32$.

Answer: See above.

5

a. In the first case, $\mathcal{P}_l = \{1, \ldots, L\}$, the same pilot sequences are reused in every cell. This will lead to pilot contamination. In the second case, $\mathcal{P}_l = \{l\}$, every cell has its own exclusive set of pilots. There is no pilot contamination in this case.

b. As $M \to \infty$, we have

$$\log_2(1+\text{SINR}_{lk}^{\text{zf,ul}}) - \log_2(1+\text{SINR}_{lk}^{\text{mr,ul}}) \to \log_2\left(1 + \frac{\rho_{\text{ul}}\gamma_{lk}^{l}\eta_{lk}}{\rho_{\text{ul}} \sum\limits_{l' \in \mathcal{P}_l \setminus \{l\}} \gamma_{l'k}^{l}\eta_{l'k}}\right) - \log_2\left(1 + \frac{\rho_{\text{ul}}\gamma_{lk}^{l}\eta_{lk}}{\rho_{\text{ul}} \sum\limits_{l' \in \mathcal{P}_l \setminus \{l\}} \gamma_{l'k}^{l}\eta_{l'k}}\right) = 0.$$

c. In this case, the summations $\sum\limits_{l' \in \mathcal{P}_l \setminus \{l\}}$ disappear since $\mathcal{P}_l \setminus \{l\}$ is an empty set. As $M \to \infty$, we have

$$\log_2(1+\text{SINR}_{lk}^{\text{zf,ul}}) - \log_2(1+\text{SINR}_{lk}^{\text{mr,ul}}) = \log_2\left(\frac{1+\text{SINR}_{lk}^{\text{zf,ul}}}{1+\text{SINR}_{lk}^{\text{mr,ul}}}\right)$$

$$\to \log_2\left(\frac{1 + \rho_{\text{ul}} \sum\limits_{l' \in \mathcal{P}_l} \sum\limits_{k'=1}^{K} \beta_{l'k'}^{l}\eta_{l'k'} + \rho_{\text{ul}} \sum\limits_{l' \notin \mathcal{P}_l} \sum\limits_{k'=1}^{K} \beta_{l'k'}^{l}\eta_{l'k'}}{1 + \rho_{\text{ul}} \sum\limits_{l' \in \mathcal{P}_l} \sum\limits_{k'=1}^{K} (\beta_{l'k'}^{l} - \gamma_{l'k'}^{l})\eta_{l'k'} + \rho_{\text{ul}} \sum\limits_{l' \notin \mathcal{P}_l} \sum\limits_{k'=1}^{K} \beta_{l'k'}^{l}\eta_{l'k'}}\right).$$

d. In the first case, $\mathcal{P}_l = \{1, \ldots, L\}$, zero-forcing and maximum-ratio are equally good asymptotically. In the second case, the limit is a positive number since

$$\sum_{l' \in \mathcal{P}_l} \sum_{k'=1}^{K} \beta_{l'k'}^{l}\eta_{l'k'} > \sum_{l' \in \mathcal{P}_l} \sum_{k'=1}^{K} (\beta_{l'k'}^{l} - \gamma_{l'k'}^{l})\eta_{l'k'}.$$

This implies that zero-forcing is better than maximum-ratio as $M \to \infty$. The conclusion is that zero-forcing is the best choice.

e. When there is only one cell, there is no pilot contamination. Hence, we can follow the same derivation as in Part **c** but remove all terms for $l > 1$:

$$\log_2(1+\text{SINR}_{1k}^{\text{zf,ul}}) - \log_2(1+\text{SINR}_{1k}^{\text{mr,ul}}) = \log_2\left(\frac{1+\text{SINR}_{1k}^{\text{zf,ul}}}{1+\text{SINR}_{1k}^{\text{mr,ul}}}\right)$$

$$\to \log_2\left(\frac{1 + \rho_{\text{ul}} \sum\limits_{k'=1}^{K} \beta_{1k'}^{l}\eta_{1k'}}{1 + \rho_{\text{ul}} \sum\limits_{k'=1}^{K} (\beta_{1k'}^{l} - \gamma_{1k'}^{l})\eta_{1k'}}\right).$$

which is also greater than zero. Hence, zero-forcing remains to be the better choice.

Answer: See above.