LINKÖPINGS UNIVERSITET

Exam in TSKS14 Multiple Antenna Communications

Exam code:	TEN1
Date:	2018-05-31 **Time:** 08:00-12:00
Place:	KÅRA
Teacher:	Emil Björnson, 013 - 28 67 32, emil.bjornson@liu.se
Visiting exam:	Around 9 and 10.
Administrator:	Carina Lindström, 013 - 28 44 23, carina.e.lindstrom@liu.se
Department:	ISY
Allowed aids:	Pocket calculator with empty memory. Olofsson: Tables and Formulas for Signal Theory.
Number of tasks:	5
Solutions:	Will be published after the exam on the course web page.
Result:	You get a message about your result via an automatic email from Ladok. Note that we cannot file your result if you are not registered on the course. That also means that you will not get an automated email about your result if you are not registered on the course.
Exam return:	2018-06-12, 12:45–13:00, Emil Björnson's office, Building B, top floor, corridor A between entrances 27–29. After that in the student office of Dept. of EE. (ISY), Building B, Corridor D, between Entrances 27–29, right next to Café Java.

Important: Solutions and answers must be given in English.

Grading: This exam consists of five problems. You can get up to five points from each problem. Thus, at most 25 points are available. Grade limits:

- Grade three: 12 points,
- Grade four: 16 points,
- Grade five: 20 points.

Sloppy solutions and solutions that are hard to read are subject to hard judgement, as are unreasonable answers.

1 Consider the MISO channel (5p)

$$y[m] = \boldsymbol{g}^H \boldsymbol{x}[m] + w[m],$$

where m is the time index, $\{\boldsymbol{x}[m]\}$ is the input sequence, $\{y[m]\}$ is the output sequence, and the additive noise term $\{w[m]\}$ is i.i.d. $CN(0,1)$ distributed. The input sequence must satisfy $E\{\|\boldsymbol{x}[m]\|^2\} \leq P$ and the channel $\boldsymbol{g}$ is a known and deterministic $M \times 1$ vector, where M denotes the number of transmit antennas.

a. What is the capacity of this channel? How is $\boldsymbol{x}[m]$ selected to achieve the capacity? (2p)

b. In practical implementations, we might not be able to select the distribution of $\boldsymbol{x}[m]$ in any way we like. Suppose $\boldsymbol{x}[m] = \boldsymbol{a}\tilde{x}[m]$ where $\tilde{x}[m] \sim CN(0, P)$ and $\boldsymbol{a}$ is a given unit-norm vector. Compute an achievable spectral efficiency (i.e., a non-zero lower bound on the capacity) in this case. (1p)

c. In some implementations, the beamforming is carried out using analog hardware that takes $\tilde{x}[m]$ as input and phase-shifts it differently at each antenna. Mathematically, this means that all the elements of $\boldsymbol{a}$ must have the same absolute value, but may have different phases. Determine the $\boldsymbol{a}$ that maximizes the spectral efficiency from part **b** in this case. (1p)

d. Under which condition on $\boldsymbol{g}$ is the capacity in part **a** and the spectral efficiency in part **c** equal? Given an example of a category of physical channels that satisfies this condition. (1p)

2 Write up the effective SINR for an uplink massive MIMO system with two user terminals and one base station, which uses maximum ratio processing. The expression should contain variables, not numbers, and no $\sum$-sign. (5p)

Write a text that explains what each of the four terms in the SINR represents and which insights one can get from their dependence on variables such as the transmit power, large-scale fading coefficient, number of antennas, etc. Make sure that all variables are explained.

As a last step, give values to all the variables and compute the effective SINRs using these values.

3 Consider the channel (5p)

$$y[m] = g \cdot x[m] + w[m],$$

where m is the time index, $\{x[m]\}$ is the input sequence with power P, $\{y[m]\}$ is the output sequence, and $\{w[m]\}$ is i.i.d. $CN(0,1)$ noise. The channel coefficient g is a realization of a random variable and distributed such that $|g|^2$ is uniformly distributed between 0 and 1. The value of g is only known at the receiver. Answer the following questions:

a. Derive the outage probability of this channel. Express the answer as a function of the desired rate $R \geq 0$. (2p)

b. Suppose we instead have two receive antennas and these observe independent channel realizations, each with the distribution defined above. What is the outage probability for the desired rate $R \geq 0$? (2p)

c. Suppose $R = \log_2(1 + 0.5P)$ is the desired rate. Is it better to use one or two receive antennas? (1p)

4 The capacity region of a two-user multiple-access channel (uplink) with perfect channel state information is determined by (5p)

$$
\begin{aligned}
R_1 &\leq \log_2(1 + P\|\boldsymbol{g}_1\|^2) \\
R_1 + R_2 &\leq \log_2 |\boldsymbol{I}_K + P\boldsymbol{G}^H\boldsymbol{G}| \\
R_2 &\leq \log_2(1 + P\|\boldsymbol{g}_2\|^2).
\end{aligned}
$$

where $P > 0$ is the SNR and $\boldsymbol{G} = [\boldsymbol{g}_1 \; \boldsymbol{g}_2] \in \mathbb{C}^{M \times K}$.

a. Sketch the capacity region and explain how it is determined by the equations above. (1p)

b. Derive the relation between $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ for which the capacity region is a rectangle. (2p)

c. Suppose $P = 0.1$, $\boldsymbol{g}_1 = [1,\, 1,\, \ldots]^T$, and $\boldsymbol{g}_2 = [1,\, -1,\, 1,\, -1,\, \ldots]^T$.
For which values of M is the sum rate 10 bit/s/Hz or more? (2p)

5 Consider the downlink of a single-cell massive MIMO system with M antennas at the base station, K single-antenna user terminals, i.i.d. Rayleigh fading channels, and maximum ratio processing. (5p)

a. Formulate an optimization problem where the power-control coefficients are selected to provide an equal effective SINR, $\overline{\text{SINR}}$, to all terminals and simultaneously maximize the value of $\overline{\text{SINR}}$. The power control coefficients must satisfy the constraint $\sum_{k=1}^{K} \eta_k \leq 1$. (1p)

b. Find a closed-form expression for η_k that solves the problem from **a** and the corresponding maximum values of $\overline{\text{SINR}}$. (3p)

c. Consider a two-user scenario with $\beta_1 > \beta_2$. Which of the two users will be allocated the largest fraction of the transmit power at the optimal solution? You must explain your answer mathematically and give explain the intuition behind the result. (1p)

Table of formulas

Capacity lower bound for the kth terminal in a single-cell system:

Uplink with arbitrary decoding vector:

$$E\left\{\log_2\left(1+\frac{\rho_{\text{ul}}\eta_k|\mathbf{a}_k^H\hat{\mathbf{g}}_k|^2}{\mathbf{a}_k^H\left(\sum_{\substack{k'=1\\k'\neq k}}^{K}\rho_{\text{ul}}\eta_{k'}\hat{\mathbf{g}}_{k'}\hat{\mathbf{g}}_{k'}^H+\left(\sum_{k'=1}^{K}\rho_{\text{ul}}\eta_{k'}(\beta_{k'}-\gamma_{k'})+1\right)\mathbf{I}_M\right)\mathbf{a}_k}\right)\right\}$$

Downlink with arbitrary precoding vectors:

$$\log_2\left(1+\frac{\rho_{\text{dl}}\eta_k|E\{\mathbf{g}_k^T\mathbf{a}_k\}|^2}{\sum_{k'=1}^{K}\rho_{\text{dl}}\eta_{k'}E\{|\mathbf{g}_k^T\mathbf{a}_{k'}|^2\}+1-\rho_{\text{dl}}\eta_k|E\{\mathbf{g}_k^T\mathbf{a}_k\}|^2}\right)$$

Effective SINR for the kth terminal in the lth cell in a multi-cell system:

Uplink maximum-ratio:

$$\frac{M\rho_{\text{ul}}\gamma_{lk}^{l}\eta_{lk}}{1+\rho_{\text{ul}}\sum_{l'\in\mathcal{P}_l}\sum_{k'=1}^{K}\beta_{l'k'}^{l}\eta_{l'k'}+\rho_{\text{ul}}\sum_{l'\notin\mathcal{P}_l}\sum_{k'=1}^{K}\beta_{l'k'}^{l}\eta_{l'k'}+M\rho_{\text{ul}}\sum_{l'\in\mathcal{P}_l\setminus\{l\}}\gamma_{l'k}^{l}\eta_{l'k}}$$

Downlink maximum-ratio:

$$\frac{M\rho_{\text{dl}}\gamma_{lk}^{l}\eta_{lk}}{1+\rho_{\text{dl}}\sum_{l'\in\mathcal{P}_l}\beta_{lk}^{l'}\left(\sum_{k'=1}^{K}\eta_{l'k'}\right)+\rho_{\text{dl}}\sum_{l'\notin\mathcal{P}_l}\beta_{lk}^{l'}\left(\sum_{k'=1}^{K}\eta_{l'k'}\right)+M\rho_{\text{dl}}\sum_{l'\in\mathcal{P}_l\setminus\{l\}}\gamma_{lk}^{l'}\eta_{l'k}}$$