TSKS14 Multiple Antenna Communications

Solutions for the exam 2018-05-31

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1

a. Following the derivation in Section 5.3.2 of "Fundamentals of Wireless Communications", the capacity is

$$C = \log_2(1 + P\|\boldsymbol{g}\|^2) \text{ bit/s/Hz}.$$

This capacity is achieved if $\boldsymbol{x}[m] = \frac{\boldsymbol{g}}{\|\boldsymbol{g}\|}\tilde{x}[m]$ where $\tilde{x}[m] \sim CN(0, P)$.

b. With an arbitrary $\boldsymbol{a}$, we get a SISO channel $y[m] = \boldsymbol{g}^H \boldsymbol{a}\tilde{x}[m] + w[m]$. The resulting spectral efficiency is

$$\log_2(1 + P|\boldsymbol{g}^H\boldsymbol{a}|^2) \text{ bit/s/Hz}.$$

Note that this is only the capacity if $\boldsymbol{a}$ is selected as in part **a**. That is why we generally call it a spectral efficiency or a lower bound on the capacity.

c. If we use the notation $\boldsymbol{g} = [g_1, \ldots, g_M]^T$ and $\boldsymbol{a} = [a_1, \ldots, a_M]^T$, then $|\boldsymbol{g}^H\boldsymbol{a}|^2 = |\sum_{m=1}^{M} g_m^* a_m|^2$. To make this term as large as possible, we want all the terms $g_m^* a_m$ to have the same phase so they add constructively. Moreover, we want $\|\boldsymbol{a}\|^2 = 1$. Hence, one way to select $\boldsymbol{a}$ is with

$$a_m = \frac{g_m}{\sqrt{M}|g_m|}.$$

The resulting vector is different from the matched filter, which in general assigned different magnitude to different antennas. Note that there are alternative solutions where one multiplies a_m in the expression above with a phase-shift that is the same for all m.

d. For this to happen, we need $\|\boldsymbol{g}\| = \sqrt{M}|g_m|$ which happens when all the terms in $\boldsymbol{g}$ have the same absolute value: $|g_m| = \|\boldsymbol{g}\|/\sqrt{M}$. One example of a category of physical channels that satisfies this condition is the line-of-sight channels described in Section 7.2.2 of "Fundamentals of Wireless Communications".

Answer: See above.

2

The effective SINR of terminal k (for $k = 1, 2$) is

$$\frac{M\rho_{\text{ul}}\gamma_k\eta_k}{1 + \rho_{\text{ul}}\sum_{k'=1}^{2}\beta_{k'}\eta_{k'}}$$

which can be obtained directly from the table of formulas on the last page of the exam, which is why no points are given for providing this expression. The expression contains the following terms for terminal k:

- Numerator: $M\rho_{\text{ul}}\gamma_k\eta_k$ which scales linearly with the number of antennas M, due to the coherent beamforming gain. It is naturally proportional to the transmitted signal power $\rho_{\text{ul}}\eta_k$, where ρ_{ul} is the maximum (normalized) transmit power and $\eta_k \in [0,1]$ is the power control coefficient. The term $\gamma_k = \frac{\tau_p\rho_{\text{ul}}\beta_k^2}{1+\tau_p\rho_{\text{ul}}\beta_k}$ is the mean-square of the channel estimate, where $\tau_p \geq 2$ is the pilot length. This is the effective strength of the channel after it has been degraded by the imperfect channel estimation, since $\beta_k \geq \gamma_k$.

- Noise: The term "1" represents the receiver noise variance, which has been normalized.

- Interference: $\rho_{\text{ul}}\beta_{k'}\eta_{k'}$ for $k' \neq k$ represents the interference from the other terminal. It is proportional to its transmit power $\rho_{\text{ul}}\eta_{k'}$ and its large-scale fading coefficient $\beta_{k'}$ since the interference is transmitted from the other terminal and reaches the base station through the channel from that terminal. Note that this term does not depends on M, so there is no coherent beamforming gain for this interfering signal.

- Interference: $\rho_{\text{ul}}\beta_{k'}\eta_{k'}$ for $k' = k$ represents the interference that the terminal causes to itself. This is called beamforming gain uncertainty since it is caused by the fact that the base station does not know the exact value of the channel from the terminal.

Finally, we can pick any numbers for the variables, such as $M = 100$, $\beta_1 = \beta_2 = 1$, $\tau_p = 2$, $\rho_{\text{ul}} = 1$, $\eta_1 = \eta_2 = 1$. This gives the effective SINRs $\frac{100\gamma_k\eta_k}{1+\rho_{\text{ul}}\sum_{k'=1}^{2}\beta_{k'}\eta_{k'}} = 22.222$ for $k = 1, 2$.

The numbers that are used in the last part must be reasonable in the sense that one can construct such a system, so $M$is an integer, $\tau_p \geq 2$, β_k are real-valued, etc.

Answer: See above.

3

- The outage probability is defined as (see (5.54) in "Fundamentals of Wireless Communications"):

$$p_{out}(R) = \mathbb{P}\left\{\log_2(1+|g|^2P) < R\right\}$$

  using the notation from the problem formulation. Since $|g|^2$ is uniformly distributed between 0 and 1, it holds that

$$\mathbb{P}\left\{|g|^2 < x\right\} = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 1. \end{cases}$$

  By utilizing this result, we can get

$$p_{out}(R) = \mathbb{P}\left\{|g|^2 < \frac{2^R-1}{P}\right\} = \begin{cases} \frac{2^R-1}{P} & R \in [0, \log_2(1+P)] \\ 1 & R > \log_2(1+P) \end{cases}.$$

  Note that we omitted the case $R < 0$ since $R \geq 0$ according to the problem formulation (and since only positive rates make any sense).

- With two antennas and maximum ratio combining, which is capacity achieving, the outage probability is defined as (see (5.62) in "Fundamentals of Wireless Communications"):

$$p_{out}(R) = \mathbb{P}\left\{\log_2(1+(|g_1|^2+|g_2|^2)P) < R\right\}$$

  where g_1 and g_2 denote the channels to the two antennas.

Since $|g_1|^2$ and $|g_2|^2$ are both uniformly distributed between 0 and 1 and independent, $|g_1|^2 + |g_2|^2$ has a "triangular" distribution with

$$\mathbb{P}\left\{|g_1|^2 + |g_2|^2 < x\right\} = \begin{cases} 0 & x < 0 \\ x^2/2 & x \in [0,1] \\ 1-(2-x)^2/2 & x \in [1,2] \\ 1 & x > 2. \end{cases}$$

By utilizing this result, we can get

$$p_{out}(R) = \mathbb{P}\left\{|g_1|^2 + |g_2|^2 < \frac{2^R-1}{P}\right\} = \begin{cases} \left(\frac{2^R-1}{P}\right)^2/2 & R \in [0, \log_2(1+P)] \\ 1-\left(2-\frac{2^R-1}{P}\right)^2/2 & R \in [\log_2(1+P), \log_2(1+2P)] \\ 1 & R > \log_2(1+2P). \end{cases}$$

- Having an additional receive antennas will never give worse performance, as long as we process the signals optimally (as is done when considering the capacity). In this case, we get $\frac{2^R-1}{P} = 0.5$ and therefore the outage probability is 0.5 in the first case and $0.5^2/2 = 0.125$ in the second case.

Answer:

4

a. Figure C.4 in "Fundamentals of Massive MIMO" is what should be drawn and one needs to point out which equation that gives each of the three sides (called (C.40)-(C.42) in that book).

b. For this to happen, we need

$$\log_2(1+P\|\boldsymbol{g}_1\|^2) + \log_2(1+P\|\boldsymbol{g}_2\|^2) = \log_2|\boldsymbol{I}_K + P\boldsymbol{G}^H\boldsymbol{G}|.$$

By expanding the determinant of the two-by-two matrix inside the last logarithm, we get

$$\log_2|\boldsymbol{I}_K + P\boldsymbol{G}^H\boldsymbol{G}| = \log_2\left((1+P\|\boldsymbol{g}_1\|^2)(1+P\|\boldsymbol{g}_2\|^2) - P^2|\boldsymbol{g}_2^H\boldsymbol{g}_1|^2\right)$$

where the largest value that the term $-P^2|\boldsymbol{g}_2^H\boldsymbol{g}_1|^2$ can take is zero, which is achieved when $\boldsymbol{g}_2^H\boldsymbol{g}_1 = 0$. In this case, the right hand side becomes $\log_2(1+P\|\boldsymbol{g}_1\|^2) + \log_2(1+P\|\boldsymbol{g}_2\|^2)$. Hence, $\boldsymbol{g}_2^H\boldsymbol{g}_1 = 0$ is the condition that we are looking for. It means that the two vectors are orthogonal and this condition has also been called favorable propagation in the course.

c. If M is even, then $\boldsymbol{g}_2^H\boldsymbol{g}_1 = 0$. In that case, the sum rate is $2\log_2(1+0.1M)$ and we need $M \geq (2^{10/2}-1)/0.1 = 310$ to give the required performance. This is satisfied for $M = 310, 312, \ldots$.

If M is odd, then $\boldsymbol{g}_2^H\boldsymbol{g}_1 = 1$ and the sum rate $\log_2((1+0.1M)^2 - 0.01)$ and we need $M \geq \frac{\sqrt{2^{10}+0.01}-1}{0.1} \approx 310.0016$. This is satisfied for $M = 311, 313, \ldots$.

In summary, the condition is satisfied for all $M \geq 310$.

Answer: See above.

5

This is an example of max-min fairness power control as described in Section 5.3 in "Fundamentals of Massive MIMO".

a. A general formulation is provided in (5.10) of "Fundamentals of Massive MIMO". In this case, we consider the downlink with maximum ratio processing, which leads to the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \overline{\text{SINR}} && (1) \\ \text{with respect to} \quad & \{\eta_k\}, \overline{\text{SINR}} && (2) \\ \text{subject to} \quad & \frac{M\rho_{\text{dl}}\gamma_k\eta_k}{1+\rho_{\text{dl}}\beta_k \sum\limits_{k'=1}^{K} \eta_{k'}} = \overline{\text{SINR}} && (3) \\ & \sum_{k=1}^{K} \eta_k \leq 1. && (4) \end{aligned}$$

b. The same procedure as in the downlink part of Section 5.3.1 in "Fundamentals of Massive MIMO" should be followed, but with $a_k = M\rho_{\text{dl}}\gamma_k$ and $b_k = \rho_{\text{dl}}\beta_k$. This results in

$$\eta_k = \frac{1+b_k}{a_k \sum_{k'=1}^{K} \frac{1+b_{k'}}{a_{k'}}} = \frac{1+\rho_{\text{dl}}\beta_k}{M\rho_{\text{dl}}\gamma_k \sum_{k'=1}^{K} \frac{1+\rho_{\text{dl}}\beta_{k'}}{M\rho_{\text{dl}}\gamma_{k'}}}$$

and

$$\overline{\text{SINR}} = \frac{1}{\sum_{k=1}^{K} \frac{1+b_k}{a_k}} = \frac{1}{\sum_{k=1}^{K} \frac{1+\rho_{\text{dl}}\beta_k}{M\rho_{\text{dl}}\gamma_k}}.$$

c. The optimal power-control coefficients are $\eta_1 = \frac{\frac{1+\rho_{\text{dl}}\beta_1}{M\rho_{\text{dl}}\gamma_1}}{\frac{1+\rho_{\text{dl}}\beta_1}{M\rho_{\text{dl}}\gamma_1}+\frac{1+\rho_{\text{dl}}\beta_2}{M\rho_{\text{dl}}\gamma_2}}$ and $\eta_2 = \frac{\frac{1+\rho_{\text{dl}}\beta_2}{M\rho_{\text{dl}}\gamma_2}}{\frac{1+\rho_{\text{dl}}\beta_1}{M\rho_{\text{dl}}\gamma_1}+\frac{1+\rho_{\text{dl}}\beta_2}{M\rho_{\text{dl}}\gamma_2}}$ in this case.

Hence, the terminal with the largest $\frac{1+\rho_{\text{dl}}\beta_k}{M\rho_{\text{dl}}\gamma_k}$ will be allocated the largest fraction of the transmit power. Since $\gamma_k = \frac{\tau_p\rho_{\text{ul}}\beta_k^2}{1+\tau_p\rho_{\text{ul}}\beta_k}$, we have that

$$\frac{1+\rho_{\text{dl}}\beta_k}{M\rho_{\text{dl}}\gamma_k} = \frac{(1+\rho_{\text{dl}}\beta_k)(1+\tau_p\rho_{\text{ul}}\beta_k)}{M\rho_{\text{dl}}(\tau_p\rho_{\text{ul}}\beta_k^2)} = \frac{1+\tau_p\rho_{\text{dl}}\rho_{\text{ul}}\beta_k^2+(1+\tau_p)\rho_{\text{ul}}\beta_k}{M\tau_p\rho_{\text{dl}}\rho_{\text{ul}}\beta_k^2} = \frac{1}{M\tau_p\rho_{\text{dl}}\rho_{\text{ul}}\beta_k^2} + \frac{1}{M} + \frac{(1+\tau_p)}{M\tau_p\rho_{\text{dl}}\beta_k}$$

which is a decreasing function of β_k. Hence, the terminal with the largest β_k will receive the least power. This means that the largest fraction of the power (more than 50%) is allocated to terminal 2, which is the one with the smallest β-value. The intuition is that this terminal has a smaller large-scale fading coefficient and therefore the SINR becomes smaller unless we compensate for it by allocating more power to terminal 2.

Answer: See above.