TSKS14 Multiple Antenna Communications

Solutions for the exam 2017-05-29

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1

a. Following the derivation in Section 5.3.1 of "Fundamentals of Wireless Communications", the capacity is

$$C = \log_2(1 + P\|\boldsymbol{h}\|^2) = \log_2(1 + PM) \text{ bit/s/Hz}.$$

This capacity is achieved if $x[m] \sim CN(0, P)$.

b. You need $M = (2^6 - 1)/P = 63$ antennas.

c. You need a power of $P = (2^6 - 1)/10 = 6.3$.

d. In this case, $\|\boldsymbol{h}\|^2 = 4M$, thus the capacity is $\log_2(1 + 4PM)$ bit/s/Hz.

Answer: See above.

2

The answer should provide a reasonably correct description of the capacity analysis for slow fading channels. The following things can give one point each (other correct and important statements may also give points):

- Correct explanation of the slow and fast fading concepts (i.e., one random channel realization while transmitting under slow fading, infinitely many channel realizations under fast fading)
- Correct definition of the outage probability
- Correct definition of the outage capacity
- Example of how to compute the outage probability for channel of own choice.
- Correct explanation of how the use of multiple antennas improves the outage probability and/or outage capacity.

Answer: See above.

3

The proof consists of two parts.

First, you need to prove that $\eta_1 + \eta_2 = 1$ at the optimal solution. This can be done by contradiction: Suppose at the optimal solution we have $\eta_1 + \eta_2 < 1$, then we can find an $\epsilon > 1$ such that the power control coefficients $\eta_1' = \epsilon\eta_1$ and $\eta_2' = \epsilon\eta_2$ also satisfy $\eta_1' + \eta_2' < 1$. Due to the structure of the SINR expressions, this new power allocation has increased both SINRs, hence a larger sum rate is achieved. This contradiction implies that $\eta_1 + \eta_2 = 1$ at the optimal solution.

Second, by using $\eta_1 + \eta_2 = 1$, we can write the sum rate as

$$\log_2\left(1 + \frac{M\rho_{\text{dl}}\gamma_1}{\rho_{\text{dl}}\beta_1 + 1}\eta_1\right) + \log_2\left(1 + \frac{M\rho_{\text{dl}}\gamma_2}{\rho_{\text{dl}}\beta_2 + 1}\eta_2\right). \tag{1}$$

Inserting these the given parameter values and $\eta_1 = 1 - \eta_2$, we obtain the sum rate

$$\log_2\left(1 + \frac{100}{3}(1 - \eta_2)\right) + \log_2\left(1 + \frac{3200}{45}\eta_2\right).$$

Taking the first derivative and equating to zero, we obtain

$$\frac{1}{\ln(2)}\left(\frac{\frac{3200}{45}}{1 + \frac{3200}{45}\eta_2} - \frac{\frac{100}{3}}{\frac{100}{3}(1 - \eta_2)}\right) = 0$$

from which one obtains

$$\eta_2 = \frac{1 + \frac{3}{100} - \frac{45}{3200}}{2} \approx 0.508.$$

and $\eta_1 = 1 - \eta_2 \approx 0.492$.

Note one can also use waterfilling to maximize the sum rate in (1).

Answer: See above.

4

a. In the case of $\boldsymbol{a}_{k'} = [1 \ldots 1]^T$ for $k' = 1, \ldots, K$, we get

$$E\{\boldsymbol{g}_k^T \boldsymbol{a}_k\} = E\{\boldsymbol{g}_k^T\}[1 \ldots 1]^T = 0$$
$$E\{|\boldsymbol{g}_k^T \boldsymbol{a}_{k'}|^2\} = E\left\{\left|\sum_{m=1}^{M} g_k^m\right|^2\right\} = M\beta_k.$$

The resulting capacity lower bound is zero.

b. In the printed exam, we accidentally wrote $\boldsymbol{a}_{k'} = \boldsymbol{g}_{k'}$ instead of $\boldsymbol{a}_{k'} = \boldsymbol{g}_{k'}^*$. The solution below only considers the latter case, but the correction of the exam took both cases into account.

In the case of $\boldsymbol{a}_{k'} = \boldsymbol{g}_{k'}^*$ for $k' = 1, \ldots, K$, we get

$$E\{\boldsymbol{g}_k^T \boldsymbol{g}_k^*\} = E\{\|\boldsymbol{g}_k\|^2\} = M\beta_k$$
$$E\{|\boldsymbol{g}_k^T \boldsymbol{g}_{k'}^*|^2\} = E\left\{\left|\sum_{m=1}^{M} g_k^m (g_{k'}^m)^*\right|^2\right\} = \begin{cases} M\beta_k\beta_{k'} & k \neq k' \\ (M^2 + M)\beta_k^2 & k = k'. \end{cases}$$

The resulting capacity lower bound is

$$\log_2\left(1 + \frac{\rho_{\text{dl}}\eta_k M^2\beta_k^2}{\sum_{k'=1}^{K}\rho_{\text{dl}}\eta_{k'}M\beta_k\beta_{k'} + 1}\right).$$

Answer: See above.

5

a. The uplink SINR can be computed as described in Table 5.4 of "Fundamentals of Massive MIMO", with $\eta_1 = 0.07$ and $\eta_2 = 1$. The resulting SINR value of both users is $20M/81 \approx 0.2469M$.

b. Using the value from (a), we get the equation

$$0.2469M = \frac{M\rho_{\text{dl}}}{\sum_{k=1}^{K} \frac{1}{\gamma_k} + \rho_{\text{dl}} \sum_{k=1}^{K} \frac{\beta_k}{\gamma_k}}$$

which can be simplified as

$$0.2469 \left(\sum_{k=1}^{K} \frac{1}{\gamma_k} + \rho_{\text{dl}} \sum_{k=1}^{K} \frac{\beta_k}{\gamma_k} \right) = \rho_{\text{dl}}$$

or

$$\rho_{\text{dl}} = \frac{0.2469(\sum_{k=1}^{K} \frac{1}{\gamma_k})}{(1 - 0.2469 \sum_{k=1}^{K} \frac{\beta_k}{\gamma_k})} = 1.07.$$

Using this value one can then obtain $\eta_1 = 0.2835$ and $\eta_2 = 0.7165$.

c. Let A_1 and A_2 denote the SINRs that the two users achieve in the uplink, we then want to find $\rho_{\text{dl}}\eta_1$ and $\rho_{\text{dl}}\eta_2$ such that

$$A_k = \frac{M\rho_{\text{dl}}\eta_k\gamma_k}{1 + \rho_{\text{dl}}\beta_k(\eta_1 + \eta_2)}$$

for $k = 1, 2$. This can be written as the equations

$$A_k = M\rho_{\text{dl}}\eta_k\gamma_k - A_k\rho_{\text{dl}}\beta_k(\eta_1 + \eta_2)$$

for $k = 1, 2$ and can be gathered in matrix form as

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} M\gamma_1 - A_1\beta_1 & -A_1\beta_1 \\ -A_2\beta_2 & M\gamma_2 - A_2\beta_2 \end{pmatrix} \begin{pmatrix} \rho_{\text{dl}}\eta_1 \\ \rho_{\text{dl}}\eta_2 \end{pmatrix}$$

from which we get the final expression

$$\begin{pmatrix} \rho_{\text{dl}}\eta_1 \\ \rho_{\text{dl}}\eta_2 \end{pmatrix} = \begin{pmatrix} M\gamma_1 - A_1\beta_1 & -A_1\beta_1 \\ -A_2\beta_2 & M\gamma_2 - A_2\beta_2 \end{pmatrix}^{-1} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Answer: See above.