TSKS04 Digital Communication Continuation Course

Solutions for the exam 2017-06-05

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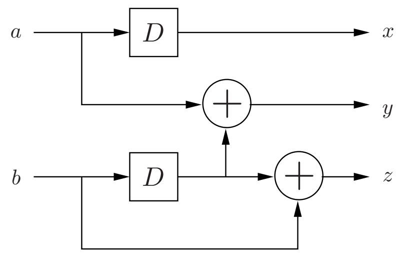
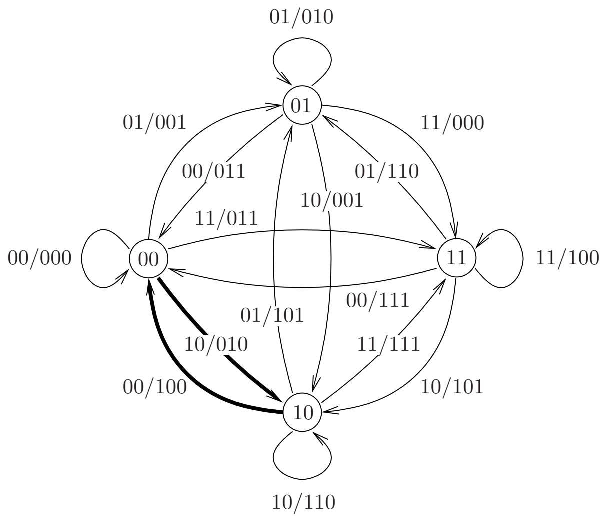
1

We were given the following encoder.



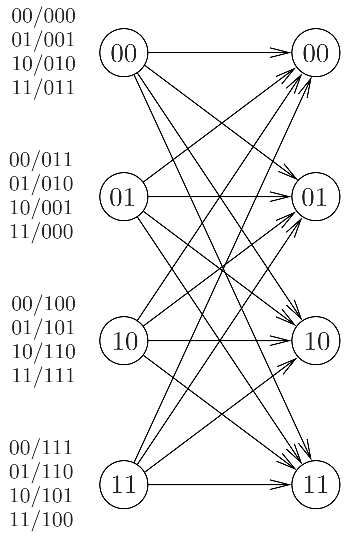
Based on the encoder, we get the following solution.

a. The states in the state diagram below are labeled with two bits. The first bit is the state of the upper register in the encoder, and the second bit is the state of the lower register. The edges are labeled $a_i b_i / x_i y_i z_i$ for time instance i.



The thicker edges will be used in part c.

b. The states and edges in the trellis section below are labeled the same way as in the state diagram above. For clarity, the labels of the edges are placed to the left of the originating state, and given in the same order as the exiting edges from that state.



c. The free distance of the code is found in the state diagram in part a by following the thicker edges. The free distance is therefore 2. Obviously, this is not a good code.

Answer: —

2

Let $p(t)$ be the total impulse response of the cascade of the sender filter and the channel i.e.

$$p(t) = (g_{\text{TX}} * g_{\text{C}})(t) = T \cdot \text{triangle}\left(\frac{t}{T} - 1\right).$$

Its matched filter is then

$$p_{\text{MF}}(t) = p^*(-t) = T \cdot \text{triangle}\left(\frac{t}{T} + 1\right).$$

Furthermore, let $y(t)$ denote the output from the channel.

a. The optimal (ML) case is if we use $p_{\text{MF}}(t)$ as our receiver filter. Let $z_0[n]$ be the output from that filter, sampled in the time instances nT. Then we have

$$z_0[n] = (y * p_{\text{MF}})(nT) = \int_{-\infty}^{\infty} y(t) p_{\text{MF}}(nT - t)\, dt.$$

The actual output from the receiver filter, sampled in the time instances $kT_{\text{s}} + \tau$ for all integers k is given by

$$z[k] = (y * g_{\text{RX}})(kT_{\text{s}} + \tau) = \int_{-\infty}^{\infty} y(t) g_{\text{RX}}(kT_{\text{s}} + \tau - t)\, dt.$$

We have

$$p_{\text{MF}}(t) = g_{\text{RX}}(t + 2T) + 2g_{\text{RX}}\left(t + \frac{3T}{2}\right) + g_{\text{RX}}(t + T),$$

which can be rewritten as

$$p_{\mathrm{MF}}(nT-t) = g_{\mathrm{RX}}\left((2n+4)\frac{T}{2}-t\right) + \\ + 2g_{\mathrm{RX}}\left((2n+3)\frac{T}{2}-t\right) + g_{\mathrm{RX}}\left((2n+2)\frac{T}{2}-t\right).$$

With $\tau = 0$ and $T_{\mathrm{s}} = T/2$ we get

$$z[k] = \int_{-\infty}^{\infty} y(t) g_{\mathrm{RX}}\left(k\frac{T}{2}-t\right)\, dt.$$

From the above, we draw the conclusion that

$$z_0[n] = z[2n+4] + 2z[2n+3] + z[2n+2]$$

holds.

b. The number of required states is M^L, where L is the memory length. The finite memory condition is $h[n] = 0, |n| > L$, where $h[n] = (p * p_{\mathrm{MF}})(nT)$. It is clear that here we have $h[n] = 0$ for $|n| > 1$, since the duration of $p(t)$ is $2T$, and it is obvious that we have $p(\pm 2T) = 0$. Thus, we have $L = 1$. The problem formulation does not specify the size of the constellation. Let M be that size. The number of needed states is then

$$M^L = M.$$

Answer:

a. ML detection is impossible for the given situation with $\tau = 0$ and $T_{\mathrm{s}} = T/2$.
b. Number of states: M, the size of the constellation.

3
Define $\Gamma = 2\pi\,\Delta f\,T$, a normalized angular frequency offset, which we introduce to make the notation more compact. Estimating Δf is then equivalent to estimating Γ. Then our samples are modeled as

$$Y[k] = b[k]e^{j\Gamma k} + W[k], \qquad k = 1, \ldots, K.$$

Define the signal $b_\Gamma[k] = b[k]e^{j\Gamma k}$ for all $k = 1, \ldots, K$, which is the received samples if there had not been any noise. Also define the vectors

$$\bar{b}_\Gamma = (b_\Gamma[1], \ldots, b_\Gamma[K]), \\ \bar{y} = \big(y[1], \ldots, y[K]\big).$$

The log likelihood ratio is given by

$$\ln\big(L(\bar{y}|\Gamma)\big) = \\ = \frac{1}{\sigma^2}\left(\mathrm{Re}\{(\bar{y}, \bar{b}_\Gamma)\} - \frac{\left|\left|\bar{b}_\Gamma\right|\right|^2}{2}\right) \\ = \frac{1}{\sigma^2}\sum_{k=1}^{K}\left[\mathrm{Re}\{y[k]b^*[k]e^{-j\Gamma k}\} - \frac{\left|b[k]e^{j\Gamma k}\right|^2}{2}\right] \\ = \frac{1}{\sigma^2}\sum_{k=1}^{K}\left[\mathrm{Re}\{y[k]b^*[k]e^{-j\Gamma k}\} - \frac{|b[k]|^2}{2}\right] \\ = \frac{1}{\sigma^2}\mathrm{Re}\{\sum_{k=1}^{K} y[k]b^*[k]e^{-j\Gamma k}\} - \frac{1}{\sigma^2}\sum_{k=1}^{K}\frac{|b[k]|^2}{2}.$$

We note that the second sum can be dropped, since all $b[k]$ are known, and thus the contribution of that term is the same regardless of Γ, and does not affect the maximization. We can also drop the multiplicative factor $\frac{1}{\sigma^2}$, since that is just a scaling of the complete expression, which also does not affect the maximization. What is left is the function

$$J(\bar{y}|\theta) = \text{Re}\{\sum_{k=1}^{K} y[k]b^*[k]e^{-j\Gamma k}\}$$

which is a suitable cost function to maximize for this situation, and that was what we were looking for. Simplify the cost function as far as you can.

Answer:

$$J(\bar{y}|\theta) = \text{Re}\{\sum_{k=1}^{K} y[k]b^*[k]e^{-j2\pi\,\Delta f\,kT}\}$$

4

We were given the two generator matrices

$$G_1(D) = \begin{pmatrix} 1 & 0 & 1+D+D^2 \\ 0 & 1 & 1+D \end{pmatrix},$$
$$G_2(D) = \begin{pmatrix} 1 & 1+D & D \\ 1 & D & 1 \end{pmatrix}.$$

a. The two generator matrices generate the same code if we can go from one of them to the other using row operations. Let $g_{i,j}(D)$ denote the j-th row of matrix $G_i(D)$. First, we note that $G_1(D)$ is on systematic form. Therefore, it is reasonable to start with $G_1(D)$ and aim at transforming it into $G_2(D)$ using row operators. First, we generate the matrix $G_3(D)$ from $G_1(D)$ by

$$g_{3,1}(D) = g_{1,1}(D) + (1+D)\cdot g_{1,2}(D) = (1, 1+D, D),$$
$$g_{3,2}(D) = g_{1,2}(D) = (0, 1, 1+D),$$

which gives us

$$G_3(D) = \begin{pmatrix} 1 & 1+D & D \\ 0 & 1 & 1+D \end{pmatrix}.$$

Next we go from $G_3(D)$ to $G_2(D)$ with the relations

$$g_{2,1}(D) = g_{3,1}(D) = (1, 1+D, D),$$
$$g_{2,2}(D) = g_{3,1}(D) + g_{3,2}(D) = (1, D, 1),$$

and as intended, we get

$$G_2(D) = \begin{pmatrix} 1 & 1+D & D \\ 1 & D & 1 \end{pmatrix}.$$

b. No! A generator matrix is catastrophic if all entries in the matrix is divisible by the same polynomial. That is not the case for any of the two matrices, since both have 1 as at least one entry.

Answer: —

5

Answer:

This is a one-dimensional situation in which the PSD is given by

$$R_s(f) = \frac{1}{T}|\Phi_1(f)|^2 R_S[fT]$$

where $\Phi_1(f)$ is the Fourier transform of the basis function $\phi_1(t)$ and $R_S[fT]$ is the PSD of the signal constellation.

The PSD can be computed as

$$R_S[fT] = \sigma_S^2 + m_S^2 \sum_m \delta(fT - m)$$

where the variance is $\sigma_S^2 = P$ according to the problem formulation. The mean value for on-off keying can be shown to be $\sqrt{P}$. This gives

$$R_S[fT] = P + P\sum_m \delta(fT - m).$$

Moreover, direction computation of Fourier transform gives

$$\Phi_1(f) = \frac{T}{2} e^{-j\pi fT} \left(\underbrace{e^{-j\pi f_c T}}_{=j^{-2f_cT}} \text{sinc}((f + f_c)T) + \underbrace{e^{+j\pi f_c T}}_{=j^{2f_cT}} \text{sinc}((f - f_c)T) \right)$$

where we can utilize that $e^{-j\pi f_c T} = j^{2f_c T}$ since $2f_cT$ is an integer. (See A.3.1 in the extra course material for details).

By multiplying everything together, according to the formula above, we get

$$\begin{aligned} R_s(f) &= \frac{T}{4} \left| e^{-j\pi fT} \left(j^{-2f_cT} \,\text{sinc}((f + f_c)T) + j^{2f_cT} \,\text{sinc}((f - f_c)T) \right) \right|^2 \left(P + P \sum_m \delta(fT - m) \right) \\ &= \frac{PT}{4} \left(\text{sinc}((f + f_c)T) + (-1)^{2f_cT} \,\text{sinc}((f - f_c)T) \right)^2 \left(1 + \sum_m \delta(fT - m) \right) \end{aligned}$$

One can continue to simplify this expression by using properties of the δ function.