TSKS04 Digital Communication Continuation Course

Solutions for the exam 2017-03-16

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Answer:

The random delay makes the signal wide-sense stationary. Since there is one basis function the PSD is given by

$$R_s(f) = \frac{1}{T}|\Phi(f)|^2 R_S[fT]$$

where $\Phi(f)$ is the Fourier transform of the basis function $\phi(t)$ and $R_S[fT]$ is the PSD of the signal constellation.

The PSD of the signal constellation can be computed as

$$R_S[fT] = \sigma_S^2 + m_S^2 \sum_m \delta(fT - m).$$

4-ASK has constellation points at $-\sqrt{E_{\max}}$, $-\sqrt{E_{\max}}/3$, $+\sqrt{E_{\max}}/3$, $+\sqrt{E_{\max}}$. Due to the symmetry around the origin we get the mean value

$$m_S = \frac{1}{4}\left(-\sqrt{E_{\max}} - \sqrt{E_{\max}}/3 + \sqrt{E_{\max}}/3 + \sqrt{E_{\max}}\right) = 0$$

and

$$\sigma_S^2 = \frac{1}{4}\left(E_{\max} + \frac{E_{\max}}{9} + \frac{E_{\max}}{9} + E_{\max}\right) = \frac{5E_{\max}}{9}.$$

Consequently, the PSD of the signal constellation is

$$R_S[fT] = \frac{5E_{\max}}{9}$$

The basis function can be interpreted as the multiplication between a pure sinusoid $\cos(2\pi f_c t)$ and a scaled rectangular box function $\mathrm{rect}(t/T)$. These have the Fourier transforms $(\delta(f - f_c) + \delta(f + f_c))/2$ and $T\,\mathrm{sinc}(Tf)$, respectively. The Fourier transform of the basis function is thus the convolution of the individual Fourier transforms, which yields

$$\Phi(f) = \frac{T}{2}\left(\mathrm{sinc}\left((f - f_c)T\right) + \mathrm{sinc}\left((f + f_c)T\right)\right).$$

By multiplying everything together, according to the formula above, we get

$$\begin{aligned} R_s(f) &= \frac{1}{T}\left|\frac{T}{2}\left(\mathrm{sinc}\left((f - f_c)T\right) + \mathrm{sinc}\left((f + f_c)T\right)\right)\right|^2 \frac{5E_{\max}}{9} \\ &= \frac{5TE_{\max}}{36}\left(\mathrm{sinc}\left((f - f_c)T\right) + \mathrm{sinc}\left((f + f_c)T\right)\right)^2 \end{aligned}$$

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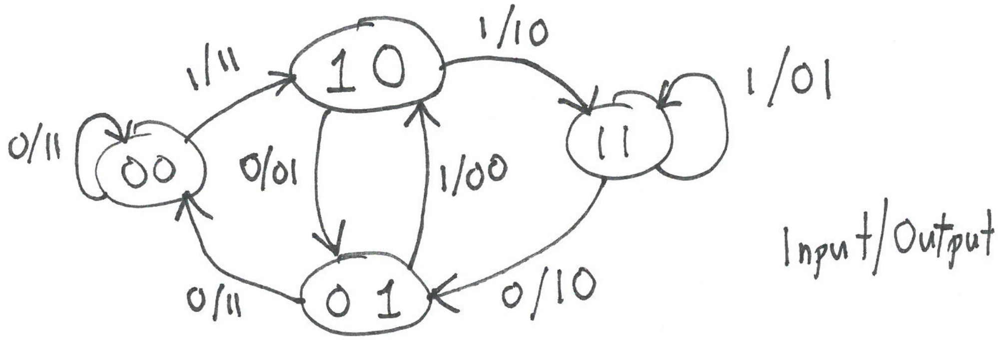
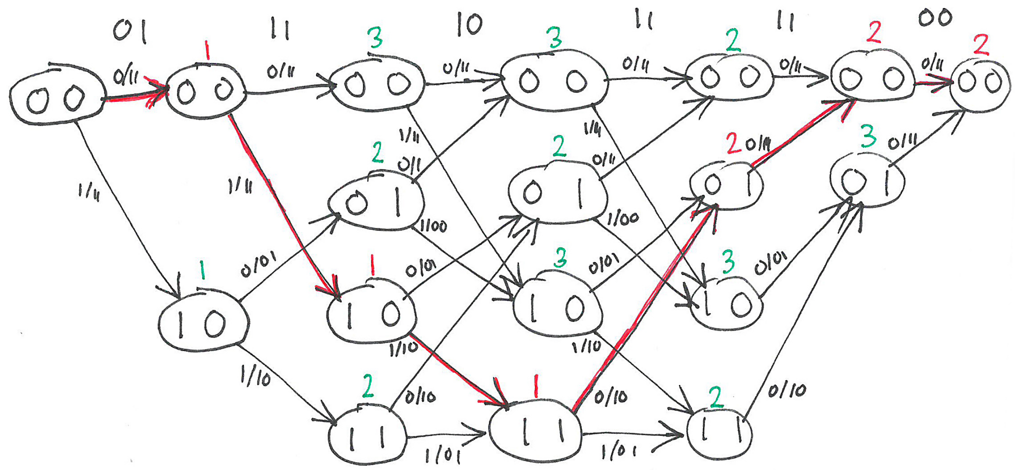
Answer:

a. The state diagram looks like this:



b. After running the Viterbi algorithm the result is:



Hence, the ML estimate of the input bits is 0 1 1 0.

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Answer:

We notice that $p(y|0) = \frac{1}{\pi}e^{-|y|^2}$ and $p(y|1) = \frac{1}{4\pi}e^{-|y|^2/4}$. By Bayes' theorem we have that

$$P[1 \text{ sent}|Y = 1 - 2j] = \frac{p(Y = 1 - 2j|1 \text{ sent })}{\frac{2}{3}p(1 - 2j|0) + \frac{1}{3}p(1 - 2j|1)}$$

Since $|1 - 2j|^2 = 5$ we have

$$P[1 \text{ sent}|Y = 1 - 2j] = \frac{\frac{1}{4\pi}e^{-5/4}}{\frac{2}{3}\frac{1}{\pi}e^{-5} + \frac{1}{3}\frac{1}{4\pi}e^{-5/4}} = \frac{1}{1 + 8e^{-15/4}}$$

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This problem is a special case of Example B.5 in the course material "Introduction to Estimation Theory for Communication". In this case, we have

$$\mathbf{y} = \begin{pmatrix} y[0] \\ \vdots \\ y[L-1] \end{pmatrix} \qquad \boldsymbol{\theta} = \begin{pmatrix} h[0] \\ \vdots \\ h[L-1] \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Answer:

a. The CRB is $(\mathbf{B}^T\mathbf{B})^{-1}$.

b. The MVU estimator $(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{y}$ attains the CRB.

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Answer:

a. One possible matrix/vector expression is $\mathbf{y} = \mathbf{S}\mathbf{h} + \mathbf{w}$ where

$$\mathbf{y} = \begin{pmatrix} y[0] \\ \vdots \\ y[N-1] \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} h[0] \\ \vdots \\ h[L-1] \end{pmatrix} \qquad \mathbf{S} = \begin{pmatrix} s[0] & 0 & \dots & 0 \\ s[1] & s[0] & 0 & \vdots \\ \vdots & \ddots & \ddots & \\ s[N-1] & s[N-2] & \dots & s[N-L] \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} w[0] \\ \vdots \\ w[N-1] \end{pmatrix}$$

b. Zero-forcing is a linear equalizer that removes inter-symbol interference. It is given by multiplying the received signal with the pseudo inverse $(\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T$ of the matrix: $(\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{y} = \mathbf{h} + (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{w}$.

c. One choice is $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ which leads to $(\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. It removes inter-user interference but the noise remains.