TSKS04 Digital Communication Continuation Course

Solutions for the exam 2016-06-07

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1

Answer:

The random delay makes the signal wide-sense stationary. Since there is one basis function the PSD is given by

$$R_s(f) = \frac{1}{T}|\Phi(f)|^2 R_S[fT]$$

where $\Phi(f)$ is the Fourier transform of the basis function $\phi(t)$ and $R_S[fT]$ is the PSD of the signal constellation.

The PSD of the signal constellation can be computed as

$$R_S[fT] = \sigma_S^2 + m_S^2 \sum_m \delta(fT - m).$$

Since we consider equally probable on-off keying we get

$$m_S = \frac{1}{2}\left(0 + \sqrt{E}\right) = \sqrt{E}/2$$

and

$$\sigma_S^2 = \frac{1}{2}\left((0 - \sqrt{E}/2)^2 + (\sqrt{E} - \sqrt{E}/2)^2\right) = E/4.$$

Consequently, the PSD of the signal constellation is

$$R_S[fT] = \frac{E}{4} + \frac{E}{4}\sum_m \delta(fT - m).$$

The basis function can be interpreted as the multiplication between a pure sinusoid $\sin(2\pi f_c t)$ and a scaled rectangular box function $\mathrm{rect}(t/T)$. These have the Fourier transforms $(\delta(f - f_c) - \delta(f + f_c))/2j$ and $T\,\mathrm{sinc}(Tf)$, respectively. The Fourier transform of the basis function is thus the convolution of the individual Fourier transforms, which yields

$$\Phi(f) = \frac{T}{2j}\left(\mathrm{sinc}\left((f - f_c)T\right) - \mathrm{sinc}\left((f + f_c)T\right)\right).$$

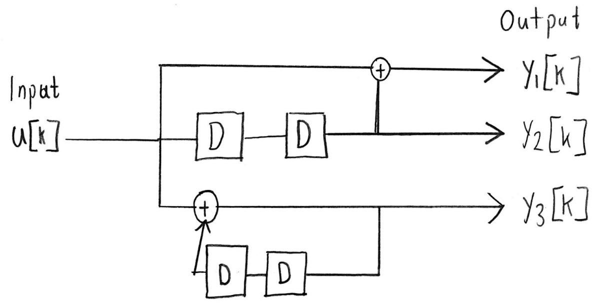
By multiplying everything together, according to the formula above, we get

$$\begin{aligned} R_s(f) &= \frac{1}{T}\left|\frac{T}{2j}\left(\mathrm{sinc}\left((f - f_c)T\right) - \mathrm{sinc}\left((f + f_c)T\right)\right)\right|^2 \left(\frac{E}{4} + \frac{E}{4}\sum_m \delta(fT - m)\right) \\ &= \frac{TE}{16}\left(\mathrm{sinc}\left((f - f_c)T\right) - \mathrm{sinc}\left((f + f_c)T\right)\right)^2 \left(1 + \sum_m \delta(fT - m)\right). \end{aligned}$$

2

Answer:

a. The encoder can look like this:



b. This input sequence can be expressed in D-transform as $U(D) = 1 + D^2$. The codeword will then be

$$U(D)G(D) = \begin{pmatrix} 1 + D^4 & D^2 + D^4 & 1 \end{pmatrix}.$$

Hence, the first output is $y_1 = (1\,0\,0\,0\,1\,0\ldots)$, the second output is $x^2 = (0\,0\,1\,0\,0\,0\ldots)$, and the third output is $x^3 = (1\,0\,0\,0\,0\,0\ldots)$.

c. No, this is not possible. One way to prove this is that the second output is a delayed version of the input, thus it has the same Hamming weight as the input. If the input sequence has infinite Hamming weight then the output has infinite Hamming weight as well.

d. Yes, this is possible. For example, the input $u = (1\,0\,0, \ldots)$ will give $y_3 = (1,\, 0,\, 1,\, 0,\, 1,\, 0,\, 1,\, \ldots)$, which has infinite Hamming weight.

3

Answer:

a. This is a signal that is observed in Gaussian noise and thus the log-likelihood function becomes

$$\begin{aligned}
\ln p(\mathbf{y}|g) &= -\frac{1}{2}\begin{pmatrix} y[1]-g & y[2]-g \end{pmatrix} \mathbf{C}^{-1} \begin{pmatrix} y[1]-g \\ y[2]-g \end{pmatrix} - \ln\left(2\pi\sqrt{|\mathbf{C}|}\right) \\
&= -\frac{1}{2(1-\rho^2)}\begin{pmatrix} y[1]-g & y[2]-g \end{pmatrix} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix} \begin{pmatrix} y[1]-g \\ y[2]-g \end{pmatrix} - \ln\left(2\pi\sqrt{1-\rho^2}\right) \\
&= -\frac{1}{2(1-\rho^2)}\left((y[1]-g)^2 + (y[2]-g)^2 - 2\rho(y[1]-g)(y[2]-g)\right) - \ln\left(2\pi\sqrt{1-\rho^2}\right).
\end{aligned}$$

b. The first derivative becomes

$$\frac{\partial}{\partial g}\ln p(\mathbf{y}|g) = -\frac{1}{2(1-\rho^2)}\left(-2(y[1]-g) - 2(y[2]-g) + 2\rho(y[1]+y[2]-2g)\right) = \frac{y[1]+y[2]-2g}{1+\rho}.$$

The regularity condition $\mathrm{E}\{\frac{\partial}{\partial g}\ln p(\mathbf{y}|g)\} = 0$ is satisfied since $\mathrm{E}\{y[1]+y[2]\} = 2g$.
We can then go ahead and compute the Fisher information

$$I(g) = -\mathrm{E}\left\{\frac{\partial^2}{\partial g^2}\ln p(\mathbf{y}|g)\right\} = \frac{2}{1+\rho}.$$

The Cramer-Rao lower bound becomes

$$\mathrm{E}\left\{(\hat{g}(\mathbf{y}) - g)^2\right\} \geq \frac{1}{I(g)} = \frac{1+\rho}{2}.$$

c. If $\rho = 1$, the matrix $\mathbf{C}$ is not invertible. The explanation is that the two observations contain exactly the same noise realization, so there is no additional information provided by second observation. If we would compute the Cramer-Rao lower bound based on only the first observation, using the same approach as above, it will become 1. This coincides with the result that we get from $\frac{1+\rho}{2}$ for $\rho = 1$.

4

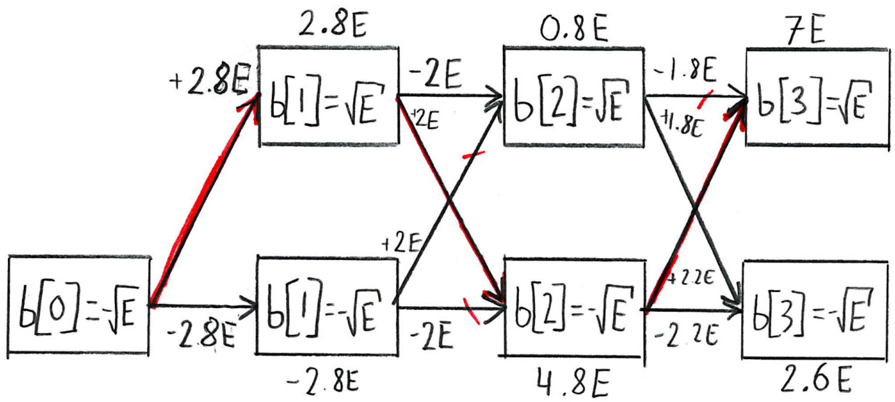
Answer:

We can utilize the Viterbi algorithm to compute the ML estimates. The branch metric in (5.13) becomes

$$\lambda_n(b[n], b[n-1]) = b[n]z[n] + |b[n]|^2 - 2b[n]b[n-1]$$

by utilizing the values of $h[m]$ that were given in the problem formulation. Since $|b[n]|^2 = E$ irrespective of which constellation point that is transmitted, we can remove this constant term from the branch metric to get $\tilde{\lambda}_n(b[n], b[n-1]) = b[n](z[n] - 2b[n-1])$.

See the finalized Viterbi algorithm below.



We terminate the algorithm by selecting the end state with the highest accumulated branch metric which is $7E$. Tracing back along the trellis gives $b[1] = \sqrt{E}$, $b[2] = -\sqrt{E}$, $b[3] = \sqrt{E}$.

5

Answer:

a. We first observe that the probability of error is the same for $b = +1$ and $b = -1$ since the noise distribution is symmetric around zero and h is independent of b. Due to this symmetry, the probability of error can be computed as

$$P_e = P\left[\hat{b} = +1 | b = -1\right] = P\left[y > 0 | b = -1\right] = P\left[y > 0 | b = -1, h = 1\right] P\left[h = 1\right] + P\left[y > 0 | b = -1, h = 2\right] P\left[h = 2\right]$$
$$= Q(1)\frac{1}{4} + Q(2)\frac{3}{4}$$

by utilizing the stated probabilities for $h = 1$ and $h = 2$ and the fact that $Q(x)$ is the probability that a standard Gaussian random variable is larger than x.

It remains to express the results in terms of E_b/N_0. We notice that the noise variance is $\sigma^2 = N_0/2 = 1$ for the considered setup, thus $N_0 = 2$. Furthermore, the average received signal energy per bit is

$$E_b = \mathbb{E}\left[(hb)^2\right] = \mathbb{E}\left[h^2\right] = 1^2\frac{1}{4} + 2^2\frac{3}{4} = \frac{13}{4}.$$

This implies that $E_b/N_0 = \frac{13}{8}$. The argument of the Q functions have the shape $c\sqrt{E_b/N_0}$, for some constant c, for communication channels of the type that we consider, where $\sqrt{E_b/N_0} = \sqrt{\frac{13}{8}}$. We can then write the probability of error as

$$P_e = Q\left(\sqrt{\frac{8}{13}}\sqrt{E_b/N_0}\right)\frac{1}{4} + Q\left(\sqrt{\frac{32}{13}}\sqrt{E_b/N_0}\right)\frac{3}{4}.$$

b. The ML detection rule gives minimum probability of error. Note that if $h = 1$ we will receive one of the two signals $+1$ or -1. If $h = 2$ we will receive one of the two signals $+2$ or -2. Since the noise is standard Gaussian the ML detection will in both cases be to select the signal point that is closest in terms of Euclidean distance. This symmetry around zero implies that $\hat{b} = \text{sign}(y)$ is the ML rule in both cases, and thus is the ML rule for any h. This proves that the statement is **true**.