# TSKS04 Digital Communication Continuation Course

## Solutions for the exam 2016-03-16

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### 1

**Answer:**

**a.** Since the information symbols are independent, the power-spectral density expression from (A.6) in Appendix A becomes

$$R_s(f) = \frac{1}{T}\sum_{i=0}^{N-1} R_{C_i}[fT]|\Phi_i(f)|^2 + R_{S_i}[fT]|\Gamma_i(f)|^2.$$

With the given BPSK modulation we also have $R_{C_i}[fT] = R_{S_i}[fT] = E$.

It remains to compute the squared Fourier transforms $|\Phi_i(f)|^2$ and $|\Gamma_i(f)|^2$. Let $f_i = f_c + \frac{i}{T}$, then we have

$$|\Phi_i(f)|^2 = \frac{T^2}{4}\Big(\mathrm{sinc}\,((f+f_i)T) + (-1)^{2f_iT}\,\mathrm{sinc}\,((f-f_i)T)\Big)^2$$

$$|\Gamma_i(f)|^2 = \frac{T^2}{4}\Big(\mathrm{sinc}\,((f+f_i)T) - (-1)^{2f_iT}\,\mathrm{sinc}\,((f-f_i)T)\Big)^2$$

where we have utilized that $2f_iT = 2f_cT + 2i$ is a positive integer when $2f_cT$ is a positive integer.

This leads to the power-spectral density expression

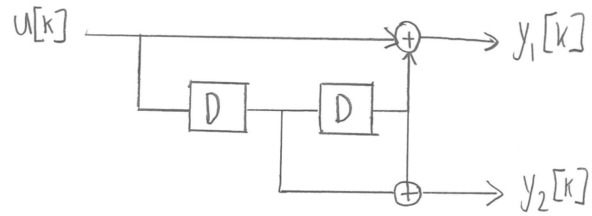
$$R_s(f) = \frac{ET}{4}\sum_{i=0}^{N-1}\left(\Big(\mathrm{sinc}\,((f+f_i)T) + (-1)^{2f_iT}\,\mathrm{sinc}\,((f-f_i)T)\Big)^2 + \Big(\mathrm{sinc}\,((f+f_i)T) - (-1)^{2f_iT}\,\mathrm{sinc}\,((f-f_i)T)\Big)^2\right).$$

**b.** There are many possible ways to approximate the bandwidth. Let us assume that $|\Phi_i(f)|^2$ and $|\Gamma_i(f)|^2$ have the approximate bandwidths $2/T$ which matches the interval length between the first zero-crossing to the left of $f_i$ and the first zero-crossing to the right of $f_i$. (See Figure A.1 in Appendix A). Since the distance between $f_i$ and $f_{i+1}$ is only $1/T$ the bandwidth of two adjacent basis functions overlap by 50 %. Consequently, the total (approximate) bandwidth of all $N$ subcarriers is $(N+1)/T$.

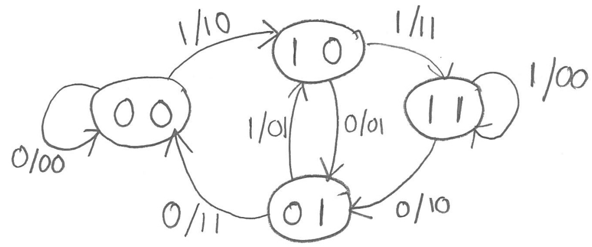
### 2

**Answer:**

**a.** The encoder can look like this:




**b**. The state-diagram can look like this:




**c**. The free distance can be computed as the minimum Hamming weight of a codeword created by leaving state 00 in the state-diagram and finding a route back to state 00 again. The route 00 → 10 → 01 → 00 creates the output sequence 10 01 11 with Hamming weight 4. This is the free distance since all other routes have larger weights.

**3**

**Answer:**

**a**. This is a signal that is observed in Gaussian noise and thus the log-likelihood function becomes

$$\ln p(\mathbf{y}|f_0) = -\frac{1}{2\sigma^2}\sum_{k=1}^{K}(y[k] - A\cos(2\pi f_0 k))^2 - \frac{K}{2}\ln\left(2\pi\sigma^2\right).$$

**b**. The first derivative becomes

$$\frac{\partial}{\partial f_0}\ln p(\mathbf{y}|f_0) = -\frac{1}{\sigma^2}\sum_{k=1}^{K}(y[k] - A\cos(2\pi f_0 k))\,A\sin(2\pi f_0 k)2\pi k.$$

The regularity condition $\mathrm{E}\{\frac{\partial}{\partial f_0}\ln p(\mathbf{y}|f_0)\} = 0$ is satisfied since $\mathrm{E}\{y[k]\} = A\cos(2\pi f_0 k)$. We can then go ahead and compute the Fisher information

$$\begin{aligned} I(f_0) &= -\mathrm{E}\left\{\frac{\partial^2}{\partial f_0^2}\ln p(\mathbf{y}|f_0)\right\} \\ &= \mathrm{E}\left\{\frac{1}{\sigma^2}\sum_{k=1}^{K}(y[k] - A\cos(2\pi f_0 k))\,A\cos(2\pi f_0 k)(2\pi k)^2 + \frac{1}{\sigma^2}\sum_{k=1}^{K}(A\sin(2\pi f_0 k)2\pi k)^2\right\} \\ &= \frac{1}{\sigma^2}\sum_{k=1}^{K}(A\sin(2\pi f_0 k)2\pi k)^2. \end{aligned}$$

The Cramer-Rao lower bound becomes

$$\mathrm{E}\left\{(\hat{f}_0(\mathbf{y}) - f_0)^2\right\} \geq \frac{1}{I(f_0)} = \frac{\sigma^2}{\sum_{k=1}^{K} \left(A \sin(2\pi f_0 k) 2\pi k\right)^2}.$$

**c.** If $2\pi f_0$ is an integer multiple of $\pi$, then $\sin(2\pi f_0 k) = 0$ and the Cramer-Rao lower bound is infinity for any $K$. This is because we sample the signal in such a way that we only observe noise. Otherwise, most of the terms $\sin^2(2\pi f_0 k)$ will be strictly positive and the summation $\sum_{k=1}^{K} \left(A \sin(2\pi f_0 k) 2\pi k\right)^2$ will grow towards infinity as $K \to \infty$. Hence, the Cramer-Rao lower bound goes to zero as $K \to \infty$ in these cases. More observations of the signal will lead to a better and better estimate of the frequency.

**4**

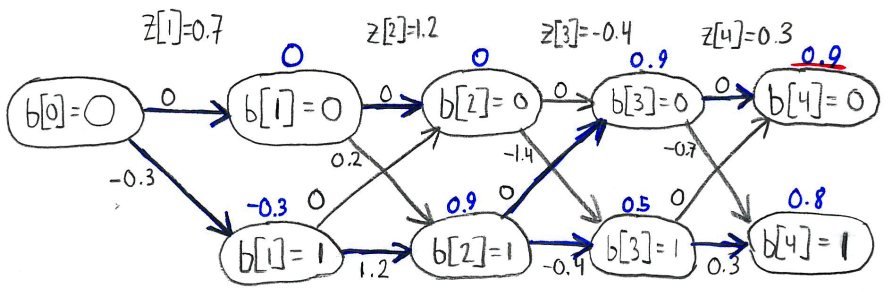
**Answer:**

We can utilize the Viterbi algorithm to compute the ML estimates. The branch metric in (5.13) becomes

$$\lambda_n(b[n], b[n-1]) = b[n]z[n] - |b[n]|^2 + b[n]b[n-1]$$

by utilizing the value of $h[m]$ that were given in the problem formulation. This can also be expressed as $\lambda_n(b[n], b[n-1]) = b[n](z[n] - b[n] + b[n-1])$ which shows that the branch metric is always 0 if $b[n] = 0$.

See the finalized Viterbi algorithm below.




When we terminate the algorithm by selecting the state with the highest accumulated branch metric which is 0.9. Tracing back along the trellis gives $b[1] = 1$, $b[2] = 1$, $b[3] = 0$, $b[4] = 0$.

**5**

**Answer:**

**a.** Using the notation from Section 5.6 in Madhow, we have

$$\mathbf{U} = \begin{pmatrix} 1.5 & 1 \\ -1.5 & 1 \\ -1 & 0 \end{pmatrix}.$$

We notice that

$$\mathbf{U}^H\mathbf{U} = \begin{pmatrix} 5.5 & 0 \\ 0 & 2 \end{pmatrix}$$

and therefore the ZF equalizer from (5.30) in Madhow is

$$\mathbf{U}(\mathbf{U}^H\mathbf{U})^{-1}\mathbf{e} = \begin{pmatrix} 1.5 & 1 \\ -1.5 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1/5.5 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}.$$

**b**. The noise covariance matrix is rank-deficient so it is possible to find correlator vectors that makes $\mathbf{c}^H \mathbf{C}_w \mathbf{c} = 0$ One of them is

$$\mathbf{c} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

which also makes $\mathbf{c}^H \mathbf{u}_0 = 1$.

**c**. Yes, it is possible to find a vector $\mathbf{c}$ that satisfies $\mathbf{c}^H \mathbf{C}_w \mathbf{c} = 0$ and $\mathbf{c}^H \mathbf{u}_{-1} = 0$. This is seen by the fact that $\mathbf{u}_0$ is not a linear combination of $\mathbf{u}_{-1}$ and $(1\ 1\ 1)^T$, where the latter is the only eigenvector of $\mathbf{C}_w$ that corresponds to a non-zero eigenvalue. The correlator vector $\mathbf{c} = (1/6\ 5/6\ \ -1)^T$ satisfies the condition.