TSKS04 Digital Communication Continuation Course

Solutions for the exam 2014-08-22

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We are given impulse responses

$$g_{\mathrm{RX}}(t) = g_{\mathrm{TX}}(t) = \mathrm{rect}\left(\frac{t}{T}\right), \quad g_{\mathrm{C}}(t) = \mathrm{rect}\left(\frac{t}{2T}\right),$$

of the receive filter, transmit filter and channel, respectively. Let $p(t)$ be the total impulse response of the cascade of the sender filter and the channel i.e.

$$p(t) = (g_{\mathrm{TX}} * g_{\mathrm{C}})(t) = \begin{cases} T, & |t| < \frac{T}{2}, \\ \frac{3T}{2} - |t|, & \frac{T}{2} \leq t < \frac{3T}{2}, \\ 0, & t > T. \end{cases}$$

We notice that $p(t)$ is real and even. Then we have the matched filter

$$p_{\mathrm{MF}}(t) = p^*(-t) = p(t).$$

Furthermore, let $y(t)$ denote the output from the channel.

a. The optimal (ML) case is if we use $p_{\mathrm{MF}}(t)$ as our receiver filter. Let $z_0[n]$ be the output from that filter, sampled in the time instances nT. Then we have

$$z_0[n] = (y * p_{\mathrm{MF}})(nT) = \int_{-\infty}^{\infty} y(t) p_{\mathrm{MF}}(nT - t)\, dt.$$

The actual output from the receiver filter, sampled in the time instances $kT_{\mathrm{s}} + \tau$ for all integers k is given by

$$z[k] = (y * g_{\mathrm{RX}})(kT_{\mathrm{s}} + \tau) = \int_{-\infty}^{\infty} y(t) g_{\mathrm{RX}}(kT_{\mathrm{s}} + \tau - t)\, dt.$$

The question is now: Can we write $p_{\mathrm{MF}}(nT - t)$ as a linear combination of $g_{\mathrm{RX}}(kT_{\mathrm{s}} + \tau - t)$ for some constants τ and T_{s} and some integer values of k? No, we cannot, at least not with a finite number of values of k and a non-zero value of T_{s}. Thus, it is not possible to perform ML sequence detection for this situation.

b. There are infinitely many choices of sender and receiver filter to make ML detection possible. The most immediate way is to keep the sender filter g_{TX} as it is, and let the receiver filter be given by

$$g_{\mathrm{RX}} = p_{\mathrm{MF}}(t)$$

as given aboove. Then ML detection is possible with $T_{\mathrm{s}} = T$ and $\tau = 0$. There will be aneed for a trellis, but that was not part of the question here.

Answer:

a. ML detection is impossible for the given situation.

b. Choose $g_{\mathrm{TX}}(t) = \begin{cases} T, & |t| < \frac{T}{2}, \\ \frac{3T}{2} - |t|, & \frac{T}{2} \leq t < \frac{3T}{2}, \\ 0, & t > T. \end{cases}$.

2

Given situation: Our cummunication is disturbed additively by N, which is Laplacian with PDF

$$f_N(n) = \frac{1}{2} e^{-|n|}.$$

For a sent A, we receive $Z = 4A + N$, and A takes values 0 and 1.

a. For the given situation, we have the conditional PDFs

$$f_{Z|A}(z|0) = f_N(z) = \frac{1}{2} e^{-|z|},$$
$$f_{Z|A}(z|1) = f_N(z-4) = \frac{1}{2} e^{-|z-4|}.$$

The log-likelihood ratio is then given by

$$K(z) = \ln\left(\frac{f_{Z|A}(z|1)}{f_{Z|A}(z|0)}\right) = |z| - |z-4| = \begin{cases} -4, & z < 0, \\ 2z - 4, & 0 \leq z < 4, \\ 4, & z \geq 4. \end{cases}$$

b. Given decision rule: Set the estimate to 1 if $z > 1$, otherwise set the estimate to 0. The probability $P_{e|1}$, i.e. the error probability given that we have sent 1 is then given by

$$\begin{aligned} P_{e|1} &= \int_{-\infty}^{1} f_{Z|A}(z|1)\,dz = \int_{-\infty}^{1} \frac{1}{2} e^{-|z-4|}\,dz \\ &= \int_{-\infty}^{1} \frac{1}{2} e^{z-4}\,dz = \left[\frac{1}{2} e^{z-4}\right]_{-\infty}^{1} \\ &= \frac{1}{2} e^{-3} \approx 0.0249. \end{aligned}$$

c. The MPE rule (minimum probability of error) as described on pages 91-92 in Madhow, states that $K(z)$ should be compared to $\ln\big(\Pr\{A{=}0\}/\Pr\{A{=}1\}\big)$. The region $z < 1$ given by decision rule can be expressed as $K(z) < -2$. The given rule is therefore MPE if we have $\ln\big(\Pr\{A{=}0\}/\Pr\{A{=}1\}\big) = -2$, which gives us the prior probability

$$\Pr\{A{=}0\} = \frac{1}{e^2+1} \approx 0.119.$$

Answer:

a. $K(z) = \begin{cases} -4, & z < 0, \\ 2z-4, & 0 \le z < 4, \\ 4, & z \ge 4. \end{cases}$

b. $P_{e|1} = \frac{1}{2} e^{-3} \approx 0.0249$

c. $\Pr\{A{=}0\} = \frac{1}{e^2+1} \approx 0.119$

3

We have a channel with output $y = hb + n$, where

- b is a the transmitted symbol, taking values ± 1.
- h is a binary random variable taking values 1 and 2 with probabilities

$$\Pr\{h{=}1\} = 1/4 \qquad \text{and} \qquad \Pr\{h{=}2\} = 3/4$$

- n is Gaussian with mean 0 and variance 1.

a. Decision rule: $\hat{b} = \operatorname{sgn}(y)$. The error probability is given by

$$\begin{aligned} P_e &= \Pr\{\hat{b} \neq b\} \\ &= \Pr\{h{=}1\}\Pr\{\hat{b} \neq b | h{=}1\} \\ &\quad + \Pr\{h{=}2\}\Pr\{\hat{b} \neq b | h{=}2\}. \end{aligned}$$

Let us analyze the two realizations of h separately. First $h = 1$:

$$\begin{aligned} &\Pr\{\hat{b} \neq b | h{=}1\} = \\ &= \Pr\{b{=}-1\}\Pr\{y{>}0 | b{=}-1, h{=}1\} \\ &\quad + \Pr\{b{=}1\}\Pr\{y{<}0 | b{=}1, h{=}1\} \\ &= \Pr\{b{=}-1\}\Pr\{n{>}1\} + \Pr\{b{=}1\}\Pr\{n{<}-1\}. \end{aligned}$$

The Gaussian variable n has an even PDF, since its expectation is 0. Therefore, we have

$$\Pr\{n{>}1\} = \Pr\{n{<}-1\}$$

and thus

$$\Pr\{\hat{b} \neq b | h{=}1\} = \Pr\{n{>}1\} = Q(1).$$

Similarily for the case $h = 2$, we get

$$\Pr\{\hat{b} \neq b | h{=}2\} = \Pr\{n{>}2\} = Q(2).$$

Putting this together, we have

$$P_e = \frac{1}{4} Q(1) + \frac{3}{4} Q(2).$$

We have

$$\begin{aligned} E_b &= \mathrm{E}\{h^2\} = \Pr\{h{=}1\} \cdot 1^2 + \Pr\{h{=}2\} \cdot 2^2 \\ &= \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot 4 = \frac{13}{4}, \end{aligned}$$

and

$$\sigma_n^2 = \frac{N_0}{2} = 1.$$

This gives us the SNR

$$\frac{E_b}{N_0} = \frac{13}{8}.$$

We know that the argument of the Q function is proportional to $\sqrt{E_b/N_0}$ for Gaussian noise. Therefore, we finally have

$$\begin{aligned} P_e &= \frac{1}{4} Q\Big(\sqrt{\frac{8}{13}}\sqrt{\frac{E_b}{N_0}}\Big) + \frac{3}{4}\Big(\sqrt{\frac{32}{13}}\sqrt{\frac{E_b}{N_0}}\Big) \\ &\approx \frac{1}{4} Q\Big(0.784\sqrt{\frac{E_b}{N_0}}\Big) + \frac{3}{4}\Big(1.569\sqrt{\frac{E_b}{N_0}}\Big) \end{aligned}$$

b. The likelihood ratio is in this case

$$L(y) = \frac{f_{Y|b}(y|1)}{f_{Y|b}(y|-1)}.$$

We have

$$\begin{aligned}
&f_{Y|b}(y|1) = \\
&= \Pr\{h{=}1\} f_{Y|b,h}(y|1,1) + \Pr\{h{=}2\} f_{Y|b,h}(y|1,2) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{4} e^{-(y-1)^2/2} + \frac{3}{4} e^{-(y-2)^2/2} \right), \\
&= \frac{e^{-y^2/2}}{4\sqrt{2\pi}} \left(e^{y-\frac{1}{2}} + 3e^{2y-2} \right), \\
&f_{Y|b}(y|-1) = \\
&= \Pr\{h{=}1\} f_{Y|b,h}(y|-1,1) + \Pr\{h{=}2\} f_{Y|-b,h}(y|1,2) \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{4} e^{-(y+1)^2/2} + \frac{3}{4} e^{-(y+2)^2/2} \right), \\
&= \frac{e^{-y^2/2}}{4\sqrt{2\pi}} \left(e^{-y-\frac{1}{2}} + 3e^{-2y-2} \right),
\end{aligned}$$

and thus the likelihood ratio

$$L(y) = \frac{e^{y-\frac{1}{2}} + 3e^{2y-2}}{e^{-y-\frac{1}{2}} + 3e^{-2y-2}}.$$

For a MPE detector, the likelihood ratio should be compared to 1. Obviously, we have $L(0) = 1$, and $L(y)$ is monotonically increasing. The given decision rule coincides with the MPE rule in this case. Thus, the answer is Yes.

Answer:

a.

$$\begin{aligned}
P_\mathrm{e} &= \frac{1}{4} Q(\sqrt{\frac{8}{13}} \sqrt{\frac{E_\mathrm{b}}{N_0}}) + \frac{3}{4} (\sqrt{\frac{32}{13}} \sqrt{\frac{E_\mathrm{b}}{N_0}}) \\
&\approx \frac{1}{4} Q(0.784 \sqrt{\frac{E_\mathrm{b}}{N_0}}) + \frac{3}{4} (1.569 \sqrt{\frac{E_\mathrm{b}}{N_0}})
\end{aligned}$$

b. Yes, the given decision rule is the MPE rule.

4

Assumption according to the proble formulation: ± 1 BPSK symbols. We are given

$$\mathrm{MMSE} = 1 - \bar{p}^\mathrm{T} \bar{c}_\mathrm{MMSE}$$

c. This demands that we solve part of problem b first: Given model (Eq. 5.25 in Madhow):

$$\bar{r}_n = b_n \bar{u}_0 + \sum_{i \neq 0} b_{n+1} \bar{u}_i + \bar{w}_n,$$

The MSE is given by

$$\mathrm{MSE} = \mathrm{E}\Big\{ \big| \langle \bar{c}, \bar{r}_n \rangle - b_n \big|^2 \Big\} = \mathrm{E}\Big\{ \big| b_n \langle \bar{c}, \bar{r}_n \rangle - 1 \big|^2 \Big\},$$

where we have used the given assumption $b_n = \pm 1$. Plugging in the model in this expression gives us

$$\begin{aligned}
\mathrm{MSE} &= \mathrm{E}\Big\{ \big| b_n \langle \bar{c}, b_n \bar{u}_0 + \sum_{i \neq 0} b_{n+1} \bar{u}_i + \bar{w}_n \rangle - 1 \big|^2 \Big\} \\
&= \mathrm{E}\Big\{ \big| \langle \bar{c}, \bar{u}_0 + b_n \sum_{i \neq 0} b_{n+1} \bar{u}_i + b_n \bar{w}_n \rangle - 1 \big|^2 \Big\}
\end{aligned}$$

Assuming that the sequence of BPSK symbols are IID and the two outcomes are equally probable, then the above simplifies to

$$\begin{aligned}
\mathrm{MSE} &= \mathrm{E}\Big\{ \big| \langle \bar{c}, \bar{u}_0 + b_n \sum_{i \neq 0} b_{n+1} \bar{u}_i + b_n \bar{w}_n \rangle - 1 \big|^2 \Big\} \\
&= \big| \langle \bar{c}, \bar{u}_0 \rangle - 1 \big|^2 + \sum_{i \neq 0} \big| \langle \bar{c}, \bar{u}_i \rangle \big|^2 + \mathrm{E}\Big\{ \big| \langle \bar{c}, \bar{w}_n \rangle \big|^2 \Big\}
\end{aligned}$$

since the expectation of all mixed terms become zero. We notice that we have $\big| \langle \bar{a}, \bar{b} \rangle \big|^2 = \bar{a}^\mathrm{T} \bar{b} \bar{b}^\mathrm{T} \bar{a}$ for vectors $\bar{a}$ and $\bar{b}$, where $\bar{b}\bar{b}^\mathrm{T}$ is a square matrix. Using that observation, we rewrite our expression as

$$\mathrm{MSE} = \big| \langle \bar{c}, \bar{u}_0 \rangle - 1 \big|^2 + \bar{c}^\mathrm{T} A \bar{c},$$

where we have $A = \sum_{i \neq 0} \bar{u}_i \bar{u}_i^\mathrm{T} + C_{\bar{w}_n}$, and where $C_{\bar{w}_n} = \mathrm{E}\{\bar{w}_n \bar{w}_n^\mathrm{T}\}$ is the correlation matrix of the noise.

We want to minimize the MSE by choosing the correlator $\bar{c}$. Therefore, we take the derivative of the MSE with respect to $\bar{c}$. This gives us

$$\begin{aligned}
\frac{d}{d\bar{c}} \mathrm{MSE} &= \frac{d}{d\bar{c}} \big| \langle \bar{c}, \bar{u}_0 \rangle - 1 \big|^2 + \bar{c}^\mathrm{T} A \bar{c}, \\
&= 2(\bar{c}^\mathrm{T} \bar{u}_0 - 1) \bar{u}_0 + 2A\bar{c}.
\end{aligned}$$

We set this equal to zero, which gives us

$$A \bar{c}_\mathrm{MMSE} = (1 - \bar{c}_\mathrm{MMSE}^\mathrm{T} \bar{u}_0) \bar{u}_0.$$

Multiplying both sides by $\bar{c}_\mathrm{MMSE}^\mathrm{T}$ gives us

$$\bar{c}_\mathrm{MMSE}^\mathrm{T} A \bar{c}_\mathrm{MMSE} = \bar{c}_\mathrm{MMSE}^\mathrm{T} \bar{u}_0 (1 - \bar{c}_\mathrm{MMSE}^\mathrm{T} \bar{u}_0)$$

According to Equation 5.42 in Madhow, we have $\bar{p} = \sigma_b^2 \bar{u}_0$. Assuming uniform distribution of b, we have $\sigma_b^2 = 1$ in this case. Then we have

$$\begin{aligned}
\mathrm{MMSE} &= 1 - \bar{p}^\mathrm{T} \bar{c}_\mathrm{MMSE} \\
&= 1 - \bar{c}_\mathrm{MMSE}^\mathrm{T} \bar{p} = 1 - \bar{c}_\mathrm{MMSE}^\mathrm{T} \bar{u}_0.
\end{aligned}$$

Since we have minimized the MSE, we have maximized the signal-to-interference ratio, which now is given by

$$\begin{aligned}\mathrm{SIR} &= \frac{(\bar{c}_{\mathrm{MMSE}}^{\mathrm{T}}\bar{u}_0)^2}{\bar{c}_{\mathrm{MMSE}}^{\mathrm{T}}\bar{u}_0(1-\bar{c}_{\mathrm{MMSE}}^{\mathrm{T}}\bar{u}_0)} = \frac{\bar{c}_{\mathrm{MMSE}}^{\mathrm{T}}\bar{u}_0}{1-\bar{c}_{\mathrm{MMSE}}^{\mathrm{T}}\bar{u}_0} \\ &= \frac{1-\mathrm{MMSE}}{\mathrm{MMSE}} = \frac{1}{\mathrm{MMSE}} - 1.\end{aligned}$$

d. From Equation 5.31, we have that the MSE for zero-forcing is given by $MSE(\bar{c}_{\mathrm{ZF}}) = \sigma_w^2 \left\|\bar{c}_{\mathrm{ZF}}\right\|^2$. Obviously, as σ_w tends to zero, so does $MSE(\bar{c}_{\mathrm{ZF}})$. But we also know that $\bar{c}_{\mathrm{MMSE}}$ minimizes the MSE. Thus, we have $\mathrm{MSE}(\bar{c}_{\mathrm{MMSE}}) \leq \mathrm{MSE}(\bar{c}_{\mathrm{ZF}})$ and consequently, as σ_w tends to zero, so does $\mathrm{MSE}(\bar{c}_{\mathrm{MMSE}})$. We have, from above,

$$\begin{aligned}&\mathrm{MSE}(\bar{c}_{\mathrm{MMSE}}) = \\ &= \left|\langle \bar{c}_{\mathrm{MMSE}}), \bar{u}_0\rangle - 1\right|^2 + \sum_{i\neq 0} \left|\langle \bar{c}_{\mathrm{MMSE}}, \bar{u}_i\rangle\right|^2 \\ &\quad + \sigma_w^2 \|\bar{c}_{\mathrm{MMSE}}\|^2\end{aligned}$$

The above is a sum of squares. Thus, since $\mathrm{MSE}(\bar{c}_{\mathrm{MMSE}})$ tends to zero as σ_w tends to zero, so must each square. This means that $\langle \bar{c}_{\mathrm{MMSE}}, \bar{u}_0\rangle \to 1$ and $\langle \bar{c}_{\mathrm{MMSE}}, \bar{u}_i\rangle \to 0$ as σ_w tends to zero. Thus, MMSE tends to ZF as $\sigma_w \to 0$.

Answer: Proven above.

5

We were given the two generator matrices

$$G_1(D) = \begin{pmatrix} 1+D & D & 1 \\ D^2 & 1 & 1+D+D^2 \end{pmatrix},$$

$$G_2(D) = \begin{pmatrix} 1 & 1+D+D^2 & D^2 \\ 1+D & D & 1 \end{pmatrix}.$$

a. The two generator matrices generate the same code if we can go from one of them to the other using row operations. Let $g_{i,j}(D)$ denote the j-th row of matrix $G_i(D)$. First, we note that $g_{1,1}(D) = g_{2,2}(D)$ holds. What is left is then to show that $g_{2,1}(D)$ can be written as a linear combination of $g_{1,1}(D)$ and $g_{1,2}(D)$. By studying the last element in those rows, we see that if there is a solution, it has to be

$$g_{2,1}(D) = (1+D)g_{1,1}(D) + g_{1,2}(D).$$

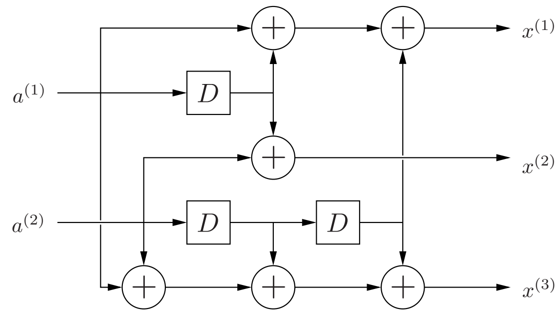
Evaluating that, we find

$$\begin{aligned}&(1+D)g_{1,1}(D) + g_{1,2}(D) = \\ &= (1+D)\big((1+D), D, 1\big) + \big(D^2, 1, 1+D+D^2\big) \\ &= \big(1+D^2, D+D^2, 1+D\big) + \big(D^2, 1, 1+D+D^2\big) \\ &= \big(1, 1+D+D^2, D^2\big),\end{aligned}$$

which indeed is $g_{2,1}(D)$. Done!

b. No! A generator matrix is catastrophic if all entries in the matrix is divisible by the same polynomial. That is not the case for any of the two matrices, since both have 1 as at least one entry.

c. An encoder, defined by $G_1(D)$:



Answer:

a. Shown above.
b. No. Both matrices are non-catastrophic.
c. See encoder above.