TSKS04 Digital Communication Continuation Course

Solutions for the exam 2014-03-17

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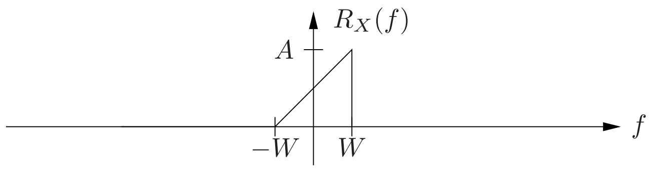
1

Since the noise and the signal are independent and since all operations are linear, we can analyze the signal and the noise separately. There are no assumptions given about the lowpass filters in the demodulation. We will for simplicity assume that the filters are ideal with amplification 1 in the passband, and whose cut-off frequency is the bandwidth W of the signal.

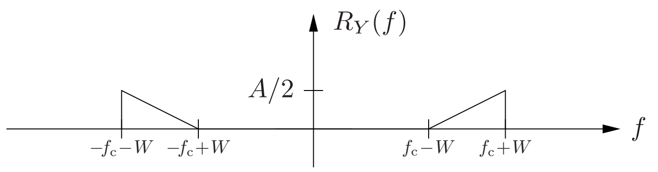
We start by analyzing the signal. We are given a complex baseband process $X(t)$ with PSD according to the following.



It is upconverted as in the left half of Figure 2.9 in Madhow. Let f_c be the carrier frequency and let $Y(t)$ be the resulting passband process. That process has PSD

$$R_Y(f) = \frac{1}{2}\big(R_X(f - f_c) + R_X(-f - f_c)\big),$$

according to equation 2.70 on page 40 in Madhow. Graphically:



Let $Z(t)$ be the signal after demodulation and filtering. According to the reasoning on pages 24-25 in Madhow, we have $Z(t) = X(t)$, where we have used the fact that the filters remove the parts around $\pm f_c$. Thus we have the PSD

$$R_Z(f) = R_X(f)$$

and the signal power

$$P_Z = P_X = \int_{-\infty}^{\infty} R_X(f)\,df = WA.$$

Next, let us analyze the noise. Let $N(t)$ be the WGN added to $Y(t)$ with PSD $N_0/2$, and let $N_1(t)$ be the resulting noise after demodulation and filtering. Then we have the noise PSD

$$R_{N_1}(f) = \begin{cases} 2R_N(f), & |f| < W, \\ 0, & \text{elsewhere,} \end{cases}$$
$$= \begin{cases} N_0, & |f| < W, \\ 0, & \text{elsewhere,} \end{cases}$$

where we have used the equation at the top of page 41 in Madhow, together with the filtering. The resulting noise signal power is

$$P_{N_1} = \int_{-\infty}^{\infty} R_{N_1}(f)\,df = 2WN_0.$$

The SNR of the output is then

$$\text{SNR} = \frac{P_Z}{P_{N_1}} = \frac{A}{2N_0}$$

Answer: $\text{SNR} = A/2N_0$

2

We have the received signal

$$y(t) = \sum_{k=1}^{M} b[k]p(t - kT)e^{j\theta} + N(t),$$

where $b[k]$ are symbols taken from a complex alphabet, and the phase θ is unknown. The noise $N(t)$ is not specified. We will assume that $N(t)$ is WGN. The receiver filters the received signal

$$z(t) = (y * p_{\text{MF}})(t)$$

where the filter is matched to $p(t)$, i.e. we have $p_{\text{MF}}(t) = p^*(-t)$. Then the result is sampled, so the rest of the receiver is supposed to have access only to $z(kT)$.

a. The signal under the hypothesis is

$$s(t) = \sum_{k=1}^{M} b[k]p(t-kT)e^{j\theta}$$

The log-likelihood function is then (since AWGN)

$$L(y|\theta,b) = \frac{1}{\sigma^2}\left(\mathrm{Re}\big\{\langle y,s\rangle\big\} - \frac{\|s\|^2}{2}\right)$$

The part of $L(y|\theta,b)$ that depends on the observed signal $y(t)$ is $\mathrm{Re}\big\{\langle y,s\rangle\big\}$. Study

$$\begin{aligned}
\langle y,s\rangle &= \int_{-\infty}^{\infty} y(t)s^*(t)\,dt \\
&= \int_{-\infty}^{\infty} y(t)\sum_{k=1}^{M} b^*[k]p^*(t-kT)e^{-j\theta}\,dt \\
&= \sum_{k=1}^{M} b^*[k]e^{-j\theta}\int_{-\infty}^{\infty} y(t)p^*(t-kT)\,dt \\
&= \sum_{k=1}^{M} b^*[k]e^{-j\theta}\int_{-\infty}^{\infty} y(t)p_{\mathrm{MF}}(kT-t)\,dt \\
&= \sum_{k=1}^{M} b^*[k]e^{-j\theta}(y*p_{\mathrm{MF}})(kT) \\
&= \sum_{k=1}^{M} b^*[k]e^{-j\theta}z(kT).
\end{aligned}$$

Thus, all the receiver needs are the samples $z(kT)$.

b. Now we have a known symbol sequence $b[k]$. We observe that $\|s\|$ does not depend on θ. The only part of $L(y|\theta,b)$ that depends on θ is $\mathrm{Re}\big\{\langle y,s\rangle\big\}$. Let $Ze^{j\phi}$ be a polar representation of $\sum_{k=1}^{M} b^*[k]z(kT)$. Then we have

$$\langle y,s\rangle = e^{-j\theta}Ze^{j\phi} = Ze^{j(\phi-\theta)},$$

and finally

$$\mathrm{Re}\big\{\langle y,s\rangle\big\} = Z\cos(\phi-\theta),$$

which is maximized by

$$\hat{\theta}_{\mathrm{ML}} = \theta = \arg\left\{\sum_{k=1}^{M} b^*[k]z(kT)\right\}.$$

Answer:

a. Shown above.
b. $\hat{\theta}_{\mathrm{ML}} = \arg\big\{\sum_{k=1}^{M} b^*[k]z(kT)\big\}$.

3

This is a variant of Task 5 from Tutorial 1.

The input process is an infinite sequence of independent stochastic variables, each having eight equally probable symbols. This input process is mapped onto two component processes, $S_1[n]$ and $S_2[n]$, one for each dimension of the signal constellation. Thus, the sample space of both $S_1[n]$ and $S_2[n]$ are $\{-A,0,A\}$, with probabilities $3/8, 2/8, 3/8$, respectively.

The PSD of the output process $S(t)$ is given by

$$R_S(f) = \frac{1}{T}\left(\Phi_1^*(f),\Phi_2^*(f)\right)\begin{pmatrix} R_{S_1}[fT] & R_{S_1,S_2}[fT] \\ R_{S_2,S_1}[fT] & R_{S_2}[fT]\end{pmatrix}\begin{pmatrix}\Phi_1(f)\\ \Phi_2(f)\end{pmatrix},$$

where $\Phi_i(f)$ is the Fourier transform of the basis function $\phi_i(t)$. Thus, we first need to determine the auto-correlation and cross-correlation functions in order to Fourier transform them into PSDs and cross-spectra.

We wish to determine the auto-correlation and cross-correlation functions

$$\begin{aligned}
r_{S_1}[k] &= \mathrm{E}\big\{S_1[n+k]S_1[n]\big\}, \\
r_{S_2}[k] &= \mathrm{E}\big\{S_2[n+k]S_2[n]\big\}, \\
r_{S_1,S_2}[k] &= \mathrm{E}\big\{S_1[n+k]S_2[n]\big\} = r_{S_2,S_1}[-k].
\end{aligned}$$

Since subsequent symbols are independent, we can identify two cases. One is $k=0$ and the other is $k\neq 0$. For $k=0$ we have

$$\begin{aligned}
r_{S_1}[0] = r_{S_2}[0] &= \mathrm{E}\big\{S_1^2[n]\big\} \\
&= \frac{1}{8}\big(3A^2 + 2\cdot 0^2 + 3(-A)^2\big) = \frac{3}{4}A^2, \\
r_{S_1,S_2}[0] &= \mathrm{E}\big\{S_1[n]S_2[n]\big\} = /\text{only non-zero terms}/ \\
&= \frac{1}{8}\big(A^2 + (-A)\cdot A + A\cdot(-A) + (-A)^2\big) = 0.
\end{aligned}$$

For $k\neq 0$ we have

$$\begin{aligned}
r_{S_1}[k] &= \mathrm{E}\big\{S_1[n+k]\big\}\mathrm{E}\big\{S_1[n]\big\}, \\
r_{S_2}[k] &= \mathrm{E}\big\{S_2[n+k]\big\}\mathrm{E}\big\{S_2[n]\big\}, \\
r_{S_1,S_2}[k] &= \mathrm{E}\big\{S_1[n+k]\big\}\mathrm{E}\big\{S_2[n]\big\},
\end{aligned}$$

where we have used the independence. Obviously, we need $\mathrm{E}\big\{S_1[n]\big\}$ and $\mathrm{E}\big\{S_2[n]\big\}$. For those, we have

$$\mathrm{E}\big\{S_1[n]\big\} = \mathrm{E}\big\{S_2[n]\big\} = \frac{1}{8}\big(3A + 2\cdot 0 + 3(-A)\big) = 0.$$

Totally, we have the correlation functions

$$r_{S_1}[k] = r_{S_2}[k] = \frac{3}{4}A^2\delta[k], \quad r_{S_1,S_2}[k] = r_{S_2,S_1}[-k] = 0,$$

and the corresponding spectra

$$R_{S_1}[\theta] = R_{S_2}[\theta] = \frac{3}{4}A^2, \quad R_{S_1,S_2}[\theta] = R^*_{S_2,S_1}[\theta] = 0.$$

These observations give us the following simplified expression of the resulting PSD:

$$R_S(f) = \frac{1}{T} R_{S_1}[fT]\Big(\big|\Phi_1(f)\big|^2 + \big|\Phi_2(f)\big|^2\Big)$$

As we can see, we need the Fourier transforms of the basis functions. Using standard properties of the Fourier transform, we find that we have

$$\begin{aligned}\Phi_1(f) &= \sqrt{T}\,\mathrm{sinc}(fT),\\ \Phi_2(f) &= \sqrt{2T}\,\mathrm{sinc}(2fT)\big(e^{j2\pi fT} - e^{-j2\pi fT}\big)\\ &= j\sqrt{8T}\,\mathrm{sinc}(2fT)\sin(2\pi fT).\end{aligned}$$

But, we are interested in the corresponding energy spectra. Those we find as

$$\begin{aligned}\big|\Phi_1(f)\big|^2 &= T\,\mathrm{sinc}^2(fT),\\ \big|\Phi_2(f)\big|^2 &= 8T\,\mathrm{sinc}^2(2fT)\sin^2(2\pi fT).\end{aligned}$$

Combining everything, we get

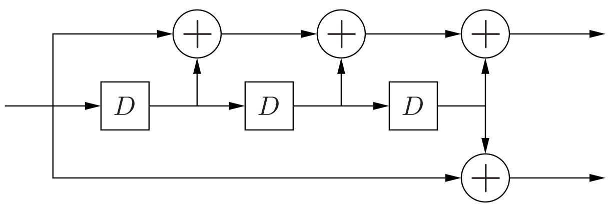
$$R_S(f) = \frac{3}{4}A^2\Big(\mathrm{sinc}^2(fT) + 8\,\mathrm{sinc}^2(2fT)\sin^2(2\pi fT)\Big).$$

Answer:

$$R_S(f) = \frac{3}{4}A^2\Big(\mathrm{sinc}^2(fT) + 8\,\mathrm{sinc}^2(2fT)\sin^2(2\pi fT)\Big).$$

4

We were given the following convolutional encoder.



a. The corresponding generator matrix is

$$G(D) = \big(1 + D + D^2 + D^3, 1 + D^3\big)$$

Both polynomials have an even weight, which means that 1 is root of both polynomials. This can also be checked by simpy plugging in $D = 1$ and finding the result 0. More precisely,

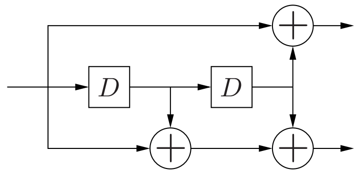
$$G(D) = (1 + D)\big(1 + D^2, 1 + D + D^2\big).$$

Thus, $G(D)$ is catastrophic.

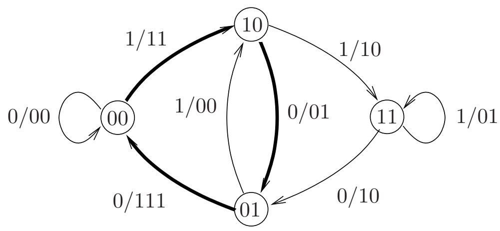
b. There are no other common factors than the one we found in part a. Dividing both polynomials with the common factor $1 + D$ gives us the non-catastrophic generator matrix

$$G_1(D) = \big(1 + D^2, 1 + D + D^2\big),$$

which we recognize as our favourite example, with the following encoder.



c. To determine the free distance, we start by drawing the state diagram of the encoder, where we label the edges by $a_i/x_i^{(1)}x_i^{(2)}$.



The path corresponding to a min-weight codeword is emphasized with bold edges. Convolutional codes are linear. Thus, the free distance is equal to the smallest non-zero weight. We count to five ones along the bold path. Therefore the code has free distance $d_{\mathrm{free}} = 5$.

Answer:
- **a.** Shown above.
- **b.** See encoder above.
- **c.** $d_{\mathrm{free}} = 5$

5

We are given impulse responses

$$\begin{aligned}g_{\mathrm{TX}}(t) &= \mathrm{rect}\left(\frac{t}{T/2}\right),\\ g_{\mathrm{C}}(t) &= \mathrm{rect}\left(\frac{t}{3T/2}\right),\\ g_{\mathrm{RX}}(t) &= \mathrm{triangle}\left(\frac{t}{T/2}\right),\end{aligned}$$

of the transmit filter, channel and receive filter, respectively. Let $p(t)$ be the total impulse response of the cascade of the sender filter and the channel i.e.

$$p(t) = (g_{\text{TX}} * g_{\text{C}})(t) = \begin{cases} 0, & t < -T, \\ (2T+t)/4, & -T \leq t < -T/2, \\ T/2, & |t| \leq T/2, \\ (2T-t)/4, & T/2 < t \leq T, \\ 0, & t > T. \end{cases}$$

We notice that $p(t)$ is real and even. Then we have the matched filter

$$p_{\text{MF}}(t) = p^*(-t) = p(t).$$

Furthermore, let $y(t)$ denote the output from the channel.

a. The optimal (ML) case is if we use $p_{\text{MF}}(t)$ as our receiver filter. Let $z_0[n]$ be the output from that filter, sampled in the time instances nT. Then we have

$$z_0[n] = (y * p_{\text{MF}})(nT) = \int_{-\infty}^{\infty} y(t) p_{\text{MF}}(nT - t)\, dt.$$

The actual output from the receiver filter, sampled in the time instances $kT_s + \tau$ for all integers k is given by

$$z[k] = (y * g_{\text{RX}})(kT_s + \tau) = \int_{-\infty}^{\infty} y(t) g_{\text{RX}}(kT_s + \tau - t)\, dt.$$

We have

$$p_{\text{MF}}(t) = \frac{T}{2}\left(g_{\text{RX}}(t + T/2) + g_{\text{RX}}(t) + g_{\text{RX}}(t - T/2)\right),$$

which can be rewritten as

$$p_{\text{MF}}(nT - t) = \frac{T}{2}\Bigg(g_{\text{RX}}\left((2n+1)\frac{T}{2} - t\right) + \\ + g_{\text{RX}}\left(2n\frac{T}{2} - t\right) + g_{\text{RX}}\left((2n-1)\frac{T}{2} - t\right)\Bigg).$$

With $\tau = 0$ and $T_s = T/2$ we get

$$z[k] = \int_{-\infty}^{\infty} y(t) g_{\text{RX}}\left(k\frac{T}{2} - t\right)\, dt.$$

From the above, we draw the conclusion that

$$z_0[n] = \frac{T}{2}\big(z[2n+1] + z[2n] + z[2n-1]\big)$$

holds.

b. The number of required states is M^L, where $M = 8$ is the size of the constellation, and where L is the memory length. The finite memory condition is $h[n] = 0, |n| > L$, where $h[n] = (p * p_{\text{MF}})(nT)$. It is clear that here we have $h[n] = 0$ for $|n| > 1$, since the duration of $p(t)$ is $2T$, and it is obvious that $h[\pm 1]$ is non-zero. Thus, we have $L = 1$. The number of needed states is then

$$M^L = 8.$$

Answer:

a. ML detection is possible for the given situation with $\tau = 0$ and $T_s = T/2$.

b. Number of states: 8