Exam in TSKS04 Digital Communication Continuation Course

Exam Code: TEN1

Date & Time: 14:00 - 18:00, 3 June, 2013

Place: U11, Hus C

Teacher: Mikael Olofsson, tel: 281343

Exam Visit: 14:45 and 17:00

Department: ISY

Allowed aids: Olofsson: Tables and Formulas for Signal Theory

U. Madhow: Fundamentals of Digital Communication

Grade Translation: Grades 3,4, and 5, are translated to ECTS C,B, and A

Solutions: Next day after the exam on the course web page

Grading: Maximum points: 100, Pass: > 40,

Grade 3: $41 - 57$, Grade 4: $58 - 74$, Grade 5: $75 - 100$

Important Instructions: All answers must be given in English

Please write legibly since partial points will be awarded for each question even if the final answer is incorrect

INSTITUTE OF TECHNOLOGY
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1) **Complex baseband representation for down conversion with a phase offset** (20 points)

Let the transmitted passband signal be given by $x_p(t) = \mathrm{Re}\big(x(t)e^{j2\pi f_c t}\big)$ where f_c is the carrier frequency and $x(t)$ is the complex baseband representation of the transmitted signal. Further, $x(t)$ has its Fourier transform band-limited to $[-W\,,\,W]$ (assume that $f_c > W$).

Consider an ideal channel where the transmitted signal is received as it is, i.e., the received passband signal is given by $y_p(t) = x_p(t)$. However, the local oscillator at the receiver has a phase offset of θ radians with respect to the phase of the local oscillator used for up conversion in the transmitter.

The output of the I branch ($z^I(t)$) in the receiver is given by

$$\begin{aligned} y_p^I(t) &= y_p(t)\cos(2\pi f_c t + \theta) \\ z^I(t) &= g(t) \star y_p^I(t) \end{aligned} \tag{1}$$

where $\star$ denotes convolution and $g(t) = 2W\,sinc(2Wt)$. Similarly the Q branch output is given by

$$\begin{aligned} y_p^Q(t) &= -y_p(t)\sin(2\pi f_c t + \theta) \\ z^Q(t) &= g(t) \star y_p^Q(t). \end{aligned} \tag{2}$$

Show that

$$z(t) \triangleq z^I(t) + jz^Q(t) = \frac{1}{2}x(t)\,e^{-j\theta} \tag{3}$$

which corresponds to a rotation by $-\theta$ radians.

($j \triangleq \sqrt{-1}$ and $sinc(x) \triangleq \frac{\sin(\pi x)}{\pi x}$)

Solution:

Using the complex baseband representation for $x_p(t)$ we have

$$\begin{aligned}
y_p^I(t) + jy_p^Q(t) &= x_p(t)\big(\cos(2\pi f_c t + \theta) - j\sin(2\pi f_c t + \theta)\big) \\
&= x_p(t)e^{-j(2\pi f_c t + \theta)} \\
&= \mathrm{Re}\big(x(t)e^{j2\pi f_c t}\big)\, e^{-j(2\pi f_c t + \theta)} \\
&= \frac{1}{2}\big(x(t)e^{j2\pi f_c t} + x^*(t)e^{-j2\pi f_c t}\big)\, e^{-j(2\pi f_c t + \theta)} \\
&= \frac{1}{2}\big(x(t)e^{-j\theta} + x^*(t)e^{-j(4\pi f_c t + \theta)}\big). \qquad (4)
\end{aligned}$$

Finally

$$\begin{aligned}
z(t) \triangleq z^I(t) + jz^Q(t) &= \big(y_p^I(t) + jy_p^Q(t)\big) \star 2W sinc(2Wt) \\
&= \frac{1}{2}\Big(\big(x(t)e^{-j\theta}\big) \star 2W sinc(2Wt) + \big(x^*(t)e^{-j(4\pi f_c t + \theta)}\big) \star 2W sinc(2Wt)\Big). \qquad (5)
\end{aligned}$$

Since $x(t)$ is bandlimited to $[-W\,,\,W]$, $\big(x^*(t)e^{-j(4\pi f_c t+\theta)}\big)$ is bandlimited to $[-2f_c - W\,,\,-2f_c + W]$, and hence $\big(x^*(t)e^{-j(4\pi f_c t+\theta)}\big) \star sinc(2Wt) = 0$ as $sinc(2Wt)$ is bandlimited to $[-W\,,\,W]$ which does not overlap with the range $[-2f_c - W\,,\,-2f_c + W]$ (since $f_c > W$).

$g(t) = 2W sinc(2Wt)$ is the impulse response of an ideal low pass filter with Fourier transform $G(f) = 1$ for $|f| \leq W$ and $G(f) = 0$ for $|f| > W$. Since $x(t)$ is bandlimited to $[-W\,,\,W]$, it is clear that $\big(x(t)e^{-j\theta}\big) \star 2W sinc(2Wt) = x(t)e^{-j\theta}$. Using the above facts in (5) we have

$$z(t) = \frac{1}{2}x(t)e^{-j\theta}. \qquad (6)$$

2) **(Modulation)** (20 points)

a) **(Tone spacing in Multicarrier communication)** (10 points)

Consider two real-valued passband waveforms

$$\begin{aligned} s_0(t) &= \cos(2\pi f_0 t)\,,\ 0 \le t \le T \\ s_1(t) &= \cos(2\pi f_1 t)\,,\ 0 \le t \le T \end{aligned} \tag{7}$$

where $f_1 > f_0 \gg 1/T$ (lets assume $f_0, f_1 > 10^9$ Hz and $T = 10$ milli second.) The pulses will be said to be orthogonal if $\int_0^T s_0(t)s_1(t)\,dt = 0$.

Show that the minimum frequency separation such that the pulses are orthogonal is $f_1 - f_0 = \frac{1}{2T}$.

b) **(Spectral efficiency of modulation schemes)** (10 points)

Consider the following two transmission schemes.

Scheme-I: The complex baseband signal is given by

$$x(t) = \sum_{k=0}^{M-1} x[k] sinc\Big(\frac{t}{T} - k\Big) \quad 0 \le t \le MT \tag{8}$$

where $x[k] \in +1\,,\,-1$ are the information carrying BPSK symbols, and M is an integer. Assume that the information symbols are independent of each other.

Scheme-II: Consider M mutually orthogonal complex baseband waveforms $\{\phi_1(t), \cdots, \phi_M(t)\}$, i.e., $\int \phi_i(t)\phi_k^*(t)\,dt = 0$ if $i \neq k$. Also assume that these waveforms are approximately time limited to $[0\,,\,MT]$ and bandlimited to $[-1/2T\,,\,1/2T]$. There are totally M possible messages that can be communicated, and to communicate the m-th message the transmitter sends the waveform $\phi_m(t)$.

Which scheme is more spectral efficient (bandwidth efficient) and why ?

Solution:

a) The inner product between $s_0(t)$ and $s_1(t)$ is given by

$$\begin{aligned} \int_0^T s_0(t)s_1(t)\,dt &= \int_0^T \cos(2\pi f_0 t)\,\cos(2\pi f_1 t)\,dt \\ &= \frac{1}{4\pi(f_0 + f_1)} \sin\big(2\pi(f_0 + f_1)T\big) + \frac{1}{4\pi(f_0 - f_1)} \sin\big(2\pi(f_0 - f_1)T\big) \\ &= \frac{T}{2}\Big(sinc\big(2(f_0+f_1)T\big) + sinc\big(2(f_1-f_0)T\big)\Big) \end{aligned} \tag{9}$$

Since $2(f_0+f_1)T \gg 1$, $sinc\big(2(f_0+f_1)T\big) \approx 0$ (for the numerical values given in the question, $2(f_0+f_1)T > 4\cdot 10^7$). Therefore, for the inner product to be 0 it suffices to choose $(f_1 - f_0)$ such that the second term in the above equation is 0. This can happen if and only if $2(f_1 - f_0)T$ is an integer, which then implies that the smallest possible spacing between f_1 and f_0 is $1/(2T)$.

b) The bit rate of scheme-I is 1 bit every T seconds (since $x[k] \in \{+1\,,\,,\,-1\}$). The bandwidth used is $1/T$ Hz (essentially the bandwidth of the sinc pulse), and therefore the spectral efficiency of the first scheme is $\eta_1 =$ bit-rate/bandwidth used $= 1$ bps/Hz.
In the second scheme, since there are only M possible messages, the number of information bits communicated in time MT is $\log_2(M)$. Hence the bit-rate of scheme-II is $\log_2(M)/(MT)$ bits per second. The bandwidth used is the same as scheme-I, i.e., $1/T$ Hz, since the waveforms $\{\phi_m(t)\,,\, m = 1, 2, \cdots, M\}$ are bandlimited to $[-1/2T\,,\, 1/2T]$. Hence the spectral efficiency of the second scheme is given by $\eta_2 = \log_2(M)/M$.
Clearly since $\log_2(M)/M \leq 1$ for all $M > 0$, scheme-I is more spectrally efficient than scheme-II.

3) **(Detection in ISI channels)** (20 points)

Consider an ISI channel with additive noise. The complex baseband discrete-time input and output are related by

$$Y[n] = \sum_{k=0}^{1} h[k]X[n-k] \, + \, W[n] \tag{10}$$

where $Y[n]$ and $X[k]$ are the n-th discrete time output sample and the k-th information symbol respectively. $h[k]$, $k = 0, 1$ is the impulse response of the channel filter. Let $h[0] = 2$ and $h[1] = -j$.

We will assume that the sequence of random variables $\{W[n]\}$ are independent and identically distributed (i.i.d.). Assume $W[n] = W^I[n] \, + \, W^Q[n]$ to be complex valued with independent and identically distributed real and imaginary components. Let the real component i.e., $W^I[n]$ have a continuous probability density function (p.d.f.) given by $f_W(w) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-w^2/(2\sigma^2)}$.

Let the input information symbols belong to the BPSK alphabet $\mathcal{A} = \{+1\,,\,-1\}$, i.e., $X[k] \in \mathcal{A}$. Also, let the input be time limited, i.e., $X[k] = 0$ for $k < 0$ and $k > 2$. Further, the input information symbols $\{x[k]\}$ are independent of additive noise.

Derive an expression for the maximum likelihood detector (MLD) of the three input BPSK symbols given the received discrete-time samples $y[n]$, $n = 0, 1, 2, 3$**.**

What is the output of the MLD if $y[0] = j, y[1] = 1 - j, y[2] = 2 + j, y[3] = 1$ **?**

Solution: In the presence of AWGN, the maximum likelihood detector for $(x[0], x[1], x[2])$ is given by

$$(\widehat{x}[0], \widehat{x}[1], \widehat{x}[2]) = \arg\min_{(x[0],x[1],x[2])\in\mathcal{A}^3} \sum_{i=0}^{3} \left|y[i] - \sum_{k=0}^{1} h[k]x[i-k]\right|^2 \tag{11}$$

Since there are only 3 symbols, there are only eight possibilities and therefore only eight likelihood values to be evaluated (one each for a possible assignment of BPSK values of $x[0]$, $x[1]$ and $x[2]$). Since there are only a handful of possibilities we can compute and find the maximum among all the eight possible likelihoods (or equivalently minimum Euclidean distance from the received vector $(j, 1-j, 2+j, 1)$).

Defining the distance function $d(x[0], x[1], x[2]) \triangleq \sum_{i=0}^{3} \left|y[i] - \sum_{k=0}^{1} h[k]x[i-k]\right|^2$, we have

$$\begin{aligned} d(x[0], x[1], x[2]) &= |y[0] - h[0]x[0]|^2 + |y[1] - h[0]x[1] - h[1]x[0]|^2 \\ &\quad + |y[2] - h[0]x[2] - h[1]x[1]|^2 + |y[3] - h[1]x[2]|^2 \end{aligned} \tag{12}$$

where we have also used the fact that the input is time limited.

Evaluating $d(x[0], x[1], x[2])$ for all the eight possible values of $(x[0], x[1], x[2])$, we find that the minimum is attained when $x[0] = 1, x[1] = 1, x[2] = 1$. This is then the maximum likelihood estimate of the BPSK information symbols.

4) **(Noncoherent Communication)** (25 points)

Consider a bandlimited passband information bearing transmit signal given by

$$X_p(t) \;=\; \mathrm{Re}\Big(X(t)e^{j(2\pi f_c t+\theta)}\Big)\;,\;\; t\in[0\,,\,T_0] \tag{13}$$

where f_c is the carrier frequency and θ is the phase of the local oscillator (LO) in the transmitter. $X(t)$ is the complex baseband information bearing signal which is bandlimited to $[-W\,,\,W]$. Assume that $f_c \gg W$ and also that $f_c T_0 \gg 1$.

Let the received signal be given by

$$Y_p(t) = AX_p(t-\Gamma)\,+\,N_p(t) \tag{14}$$

where $A > 0$ models the magnitude of the random channel gain/loss and Γ models the delay of the single path from the transmitter to the receiver. $N_p(t)$ is the AWGN.

The receiver is equipped with a local oscillator whose phase at time t is $2\pi f_c t + \phi$. Further the receiver does not know the exact value of A, Γ, θ (The receiver of course has perfect knowledge of its own LO's phase offset ϕ).

Assume the channel parameters, i.e., A, Γ, θ change so rapidly that estimation of these unknown parameters using training is not feasible.

Formulate a generalized likelihood detector for detecting the transmitted information signal non-coherently (i.e., without explicitly estimating the channel parameters).

Given a received passband waveform $y_p(t)$, show that the generalized likelihood detector is given by

$$\Big(\widehat{X}(t)\,,\,\widehat{\Gamma}\Big) \;=\; \arg\max_{x(t)\in\mathcal{X},\tau\geq 0} \frac{|\int_{\tau}^{\tau+T_0} y_p(t)e^{j(2\pi f_c t+\phi)}x(t-\tau)\,dt\;|^2}{\int_0^{T_0}|x(t)|^2\,dt} - \frac{1}{2}\int_{\tau}^{\tau+T_0} y_p^2(t)\,dt \tag{15}$$

where $\mathcal{X}$ is the set of all complex baseband information bearing transmit signals.

Solution:

In the presence of additive white Gaussian noise (AWGN), the generalized likelihood detector (for the received passband waveform $y_p(t)$) is given by

$$\begin{aligned}
\Big(\widehat{X}(t)\,,\,\widehat{\Gamma}\,,\,\widehat{A}\,,\,\widehat{\theta}\Big) &= \arg\min_{x(t)\in\mathcal{X},\tau\geq 0,A\geq 0,\theta\in[-\pi\,,\,\pi)} \int_{\tau}^{\tau+T_0} \Big(y_p(t)\,-\,Ax_p(t-\tau)\Big)^2\,dt \\
&= \arg\min_{x(t)\in\mathcal{X},\tau\geq 0,A\geq 0,\theta\in[-\pi\,,\,\pi)} \int_{\tau}^{\tau+T_0} \Big(y_p(t)\,-\,A\,\mathrm{Re}\big(x(t-\tau)e^{j(2\pi f_c(t-\tau)+\theta)}\big)\Big)^2\,dt \\
&= \arg\min_{x(t)\in\mathcal{X},\tau\geq 0,A\geq 0,\theta\in[-\pi\,,\,\pi)} \int_{\tau}^{\tau+T_0} \Big(y_p(t)\,-\,A\,\mathrm{Re}\big(x(t-\tau)e^{j(2\pi f_c(t-\tau)+\phi+(\theta-\phi))}\big)\Big)^2\,dt
\end{aligned} \tag{16}$$

The receiver however only has the knowledge of its own LO's phase offset ϕ, and this is the reason for adding and subtracting ϕ in the last line above. Substituting $\Delta = \theta - \phi$, the generalized likelihood detector above can be equivalently stated as

$$\Big(\widehat{X}(t)\,,\,\widehat{\Gamma}\,,\,\widehat{A}\,,\,\widehat{\Delta}\Big) = \arg\min_{x(t)\in\mathcal{X},\tau\geq 0,A\geq 0,\Delta\in[-\pi\,,\,\pi)} \int_{\tau}^{\tau+T_0} \Big(y_p(t)\,-\,A\,\mathrm{Re}\big(x(t-\tau)e^{j2\pi f_c(t-\tau)+\phi+\Delta}\big)\Big)^2\,dt \tag{17}$$

Keeping $x(t), \tau, A$ fixed and optimizing only over Δ, we are able to reduce the above minimization problem to the following

$$\begin{aligned}
\Big(\widehat{X}(t)\,,\,\widehat{\Gamma}\,,\,\widehat{A}\Big) &= \arg\min_{x(t)\in\mathcal{X},\tau\geq 0,A\geq 0} \int_{\tau}^{\tau+T_0} y_p^2(t)\,dt \,+\, \frac{A^2}{2}\int_0^{T_0} |x(t)|^2\,dt \\
&\qquad -2A\left|\int_{\tau}^{\tau+T_0} y_p(t)e^{j2\pi f_c(t-\tau)+\phi}x(t-\tau)\,dt\right| \\
&= \arg\min_{x(t)\in\mathcal{X},\tau\geq 0,A\geq 0} \int_{\tau}^{\tau+T_0} y_p^2(t)\,dt \,+\, \frac{A^2}{2}\int_0^{T_0} |x(t)|^2\,dt \\
&\qquad -2A\left|\int_{\tau}^{\tau+T_0} y_p(t)e^{j2\pi f_c t+\phi}x(t-\tau)\,dt\right|
\end{aligned} \tag{18}$$

We next keep $x(t), \tau$ fixed, and optimize only over A, resulting in the generalized likelihood detector given in (15).

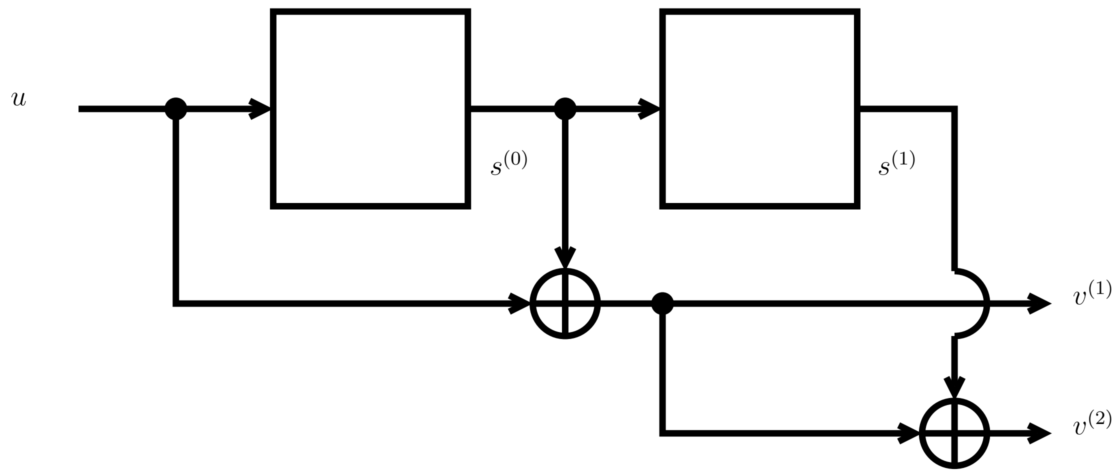

Fig. 1. Rate-1/2 binary convolutional encoder.

5) (15 points) Consider a rate $R = 1/2$ binary convolutional encoder with generator matrix

$$G(D) = \begin{bmatrix} 1 + D & 1 + D + D^2 \end{bmatrix} \tag{19}$$

as shown in Fig. 1.

Draw the state diagram for the binary convolutional encoder with generator matrix given in (19).

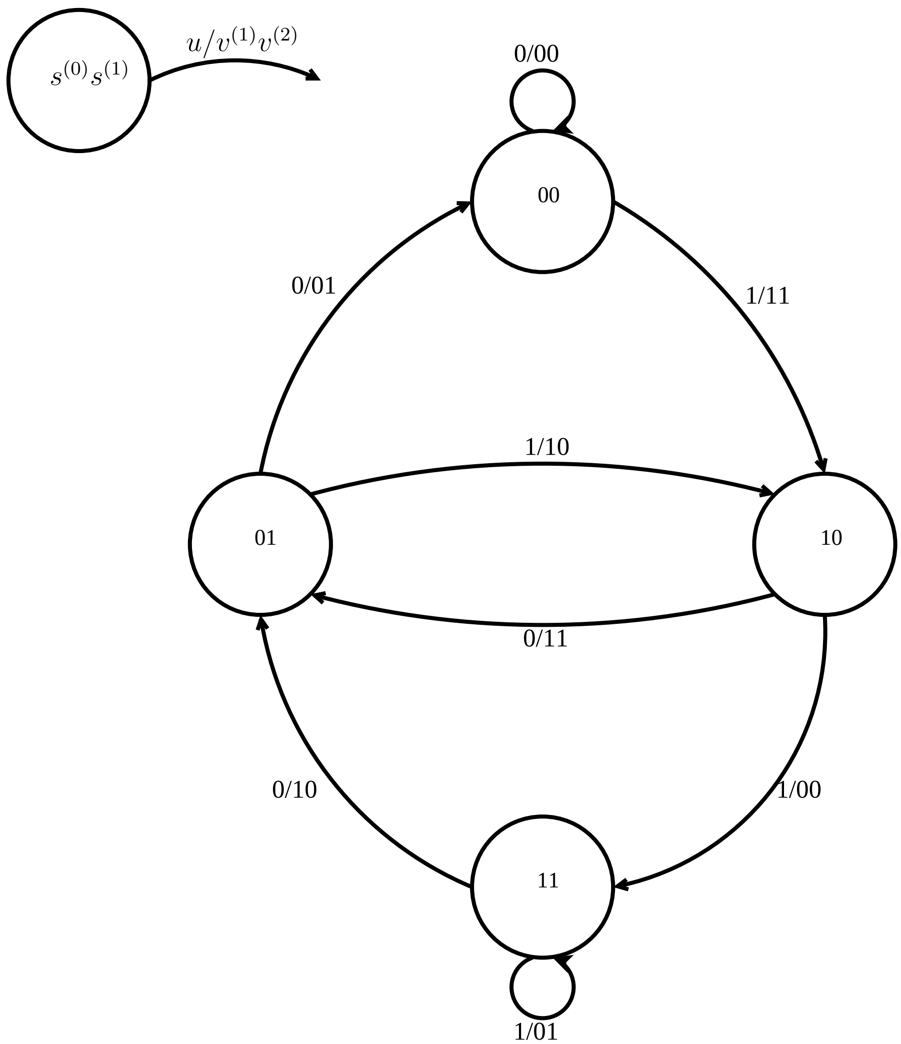

Fig. 2. State Diagram.

Solution:

The state diagram is drawn in Fig. 2.