Exam in TSKS04 Digital Communication Continuation Course

Exam Code: TEN1

Date & Time: 8:00 - 12:00, 11 March, 2013

Place: KÅRA

Teacher: Mikael Olofsson, tel: 281343

Exam Visit: Approximately 9 and 11

Administrator: Carina Lindström, tel: 284423

Department: ISY

Number of tasks: 5

Allowed aids: Olofsson: Tables and Formulas for Signal Theory

U. Madhow: Fundamentals of Digital Communication

Grade Translation: Grades 3,4, and 5, are translated to ECTS C,B, and A

Solutions: Next day after the exam on the course web page

Grading: Maximum points: 100, Pass: > 40,

Grade 3: $41-57$, Grade 4: $58-74$, Grade 5: $75-100$

Important Instructions: All answers must be given in English

Please write legibly since partial points will be awarded for each question even if the final answer is incorrect

INSTITUTE OF TECHNOLOGY
LINKÖPING UNIVERSITY

1) **Complex baseband representation for a two-path passband channel** (15 points)

Consider a linear time-invariant passband channel whose impulse response is given by $h_p(t) = \sum_{i=1}^{2} \delta(t - \tau_i)$. This models a two-path delay channel where the signal from the transmitter reaches the receiver through two different paths, with delays τ_1 and τ_2 seconds respectively.

The transmitted passband signal is given by $x_p(t) = \text{Re}\big(x(t)e^{j2\pi f_c t}\big)$ where f_c is the carrier frequency and $x(t)$ is the complex baseband representation of the transmitted signal. Further, $x(t)$ has its Fourier transform band-limited to $[-W\,,\,W]$.

Let the received passband signal and its complex baseband representation be denoted by $y_p(t)$ and $y(t)$ respectively. Therefore we have $y_p(t) = h_p(t) \star x_p(t)$, where $\star$ denotes convolution (assume that there is no additive noise).

a) **Show that**

$$y(t) = \sum_{i=1}^{2} x(t - \tau_i)\, e^{-j2\pi f_c \tau_i} \tag{1}$$

2) **Linear modulation** (20 points)

Consider linear modulation, where the band-limited complex baseband transmit signal is given by

$$x(t) = \sum_{k=-\infty}^{\infty} x[k]\, p(t - kT). \tag{2}$$

Here $\{x[k]\}$ is the sequence of information bearing symbols, and $p(t)$ is the band-limited pulse shaping filter. Assume that there is no channel filtering, i.e., channel impulse response $h_p(t) = \delta(t)$. At the receiver, the received complex baseband signal $y(t) = x(t) + n(t)$ is passed through a matching filter with impulse response $p(-t)$ (assume $p(\cdot)$ to be a real-valued waveform). The output $z(t) = y(t) \star p(-t)$ is then sampled every T seconds to result in the discrete-time samples $z[k] = z(kT)$. Also, assume that the sequence $\{x[k]\}$ is wide sense stationary and ergodic in autocorrelation.

a) $(5$ points) **What is the condition on $p(t)$ so that there is no I.S.I. (inter-symbol interference) at the receiver ?**

b) $(5$ points) **Find the Nyquist pulse having the minimum bandwidth occupancy.**

c) $(5$ points) **Show that the pulse**

$$s(t) = sinc\Big(\frac{t}{T}\Big) \frac{\cos(\frac{\pi t}{2T})}{1 - (\frac{t}{T})^2} \tag{3}$$

is Nyquist ($sinc(x) \stackrel{\Delta}{=} \sin(\pi x)/(\pi x)$). How much excess bandwidth does $s(t)$ occupy when compared to the minimum bandwidth Nyquist pulse.

d) $(5$ points) **Under what conditions is the spectral shape of the power spectral density (PSD) of $x(t)$ (in (2)) independent of the inter-symbol period T ?**

3) **(Filtering Channels)** (20 points)

Consider an ISI (filtering) channel with additive noise. The complex baseband discrete-time input and output are related by

$$Y[n] = \sum_{k=0}^{1} h[k]X[n-k] \, + \, W[n] \tag{4}$$

where $Y[n]$ and $X[k]$ are the n-th discrete time output sample and the k-th information symbol respectively. $h[k]\,,\, k = 0, 1$ is the impulse response of the channel filter. $W[n]$ is the additive noise. We will assume that the sequence of random variables $\{W[n]\}$ are independent and identically distributed (i.i.d.). Assume $W[n] = W^I[n] \, + \, jW^Q[n]$ to be complex valued with independent and identically distributed real and imaginary components. Let the real component i.e., $W^I[n]$ have a continuous probability density function (p.d.f.) denoted by $f_W(\cdot)$.
Let the input information symbols belong to some complex-valued alphabet $\mathcal{A} \subset \mathcal{C}$, i.e., $X[k] \in \mathcal{A}$. Also, let the input be time limited, i.e., $X[k] = 0$ for $k < 0$ and $k > 2$. Further, the input information symbols $\{X[k]\}$ are independent of the additive noise.
Give an expression for the maximum likelihood detector given the observed discrete time output samples $y[0], y[1], y[2], y[3]$**.**

What is maximum likelihood estimate of the transmitted information symbols if we assume that the i.i.d. real and imaginary components of the additive noise are white and Gaussian distributed with zero mean.

4) **(Estimating the unknown phase of a sinusoid of known frequency in Gaussian noise)** (15 points)

Consider, that the received passband signal is given by

$$Y_p(t) \;\; = \;\; \cos(2\pi f_c t + \Theta) \, + \, N_p(t) \quad , \quad t \in [0\,,\, T_0] \tag{5}$$

where Θ is the unknown phase and $N_p(t)$ is the AWGN. Given the received waveform $y_p(t)\,,\, t \in [0\,,\, T_0]$, the receiver needs to estimate the unkown phase offset Θ.
Write down the log-likelihood function of the received signal conditioned on the phase offset. Using this compute the maximum likelihood estimate of the unknown phase offset.
Assume that $f_c T_0 \gg 1$.

5) **(Channel Coding)** (30 points)

Consider a rate $R = 2/3$ binary convolutional encoder with generator matrix

$$G(D) = \begin{bmatrix} 1+D & D & 1+D \\ 1 & 1 & D \end{bmatrix}$$

The D-transforms of the output and the input sequences are related by the generator matrix as follows.

$$\begin{bmatrix} v_1(D) \; v_2(D) \; v_3(D) \end{bmatrix} \;=\; \begin{bmatrix} u_1(D) \; u_2(D) \end{bmatrix} G(D)$$

where $u_1(D) = \sum_n u_1(n)D^n$ and $u_2(D) = \sum_n u_2(n)D^n$ are the D-transforms of the two input sequences $\{u_1(n)\}$ and $\{u_2(n)\}$. Similarly, $v_1(D)$, $v_2(D)$ and $v_3(D)$ are the D-transforms of the three output sequences $\{v_1(n)\}$, $\{v_2(n)\}$ and $\{v_3(n)\}$ respectively.

Assume that the convolutional encoder starts in the all-zero state. Each coded output bit is mapped to a BPSK symbol ($'0'$ is mapped to $+1$, and $'1'$ is mapped to -1). Let $S[k]$ denote the k-th BPSK symbol at the output of the modulation mapper. More precisely, $S[3n] = (-1)^{v_1(n)}$, $S[3n+1] = (-1)^{v_2(n)}$ and $S[3n+2] = (-1)^{v_3(n)}$.

The BPSK symbols are then linearly-modulated to generate the complex baseband transmit signal, which is given by

$$S(t) = \sum_{k=0}^{\infty} S[k]\text{sinc}(\frac{t}{T} - k).$$

Assuming an AWGN channel (no filtering), the received discrete-time complex baseband symbols are given by

$$Y[k] \;=\; S[k] \,+\, W[k] \tag{6}$$

where $W[k] = W^I[k] + jW^Q[k]$ has independent and identically distributed (i.i.d.) Gaussian real and imaginary components (mean zero). Further the sequence of complex Gaussian random variables $\{W[k]\}$ is also i.i.d.

a) **Draw the state diagram for the convolutional encoder above.** (15 points)

b) Consider that the receiver performs hard decision decoding (i.e., hard decisions on the received samples $y[k]$, followed by maximum likelihood detection). Suppose that the transmitter sends six information bits followed by two '0' bits (for trellis termination) (information bits refers to the bits at the input of the convolutional encoder). That is the sequence of inputs to the convolutional encoder is $\left\{(u_1[n], u_2[n])\right\}_{n=0}^{n=3}$, where both the transmitter and receiver know that $(u_1[3], u_2[3]) = (0, 0)$.

Let the received discrete-time sequence be $y[0] = -2.1 + j1.7$,$y[1] = -3.2 - j0.7$,$y[2] = -1.7 + j1.1$,$y[3] = 3.8 + j2.9$, $y[4] = 0.5 - j1.6$,$y[5] = -0.6 - j2.7$, $y[6] = 0.3 + j3.2$,$y[7] = 1.6 - j0.8$, $y[8] = 1.2 - j1.3$, $y[9] = -1.0 + j0.3$, $y[10] = -0.5 + j2.3$, $y[11] = 0.3 - j0.9$.

Using the Viterbi algorithm (with hard-decisions as input) find out the transmitted information bit sequence having maximum likelihood. Assume that the receiver knows that the transmitter sends six information bits followed by '00'. (15 points)