Exam in TSKS04 Digital Communication Continuation Course

Exam Code:	TEN1
Date & Time:	8:00 - 12:00, 11 March, 2013
Place:	U11, Hus C
Teacher:	Mikael Olofsson, tel: 281343
Exam Visit:	9:00 and 11:20
Department:	ISY
Allowed aids:	Olofsson: Tables and Formulas for Signal Theory
	U. Madhow: Fundamentals of Digital Communication
Grade Translation:	Grades 3,4, and 5, are translated to ECTS C,B, and A
Solutions:	Next day after the exam on the course web page
Grading:	Maximum points: 100, Pass: > 40,
	Grade 3: $41-57$, Grade 4: $58-74$, Grade 5: $75-100$
Important Instructions:	All answers must be given in English
	Please write legibly since partial points will be awarded for each question even if the final answer is incorrect

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1) **Complex baseband representation for a two-path passband channel** (15 points)

Consider a linear time-invariant passband channel whose impulse response is given by $h_p(t) = \sum_{i=1}^{2} \delta(t - \tau_i)$. This models a two-path delay channel where the signal from the transmitter reaches the receiver through two different paths, with delays τ_1 and τ_2 seconds respectively.
The transmitted passband signal is given by $x_p(t) = \text{Re}\big(x(t)e^{j2\pi f_c t}\big)$ where f_c is the carrier frequency and $x(t)$ is the complex baseband representation of the transmitted signal. Further, $x(t)$ has its Fourier transform band-limited to $[-W\,,\,W]$.
Let the received passband signal and its complex baseband representation be denoted by $y_p(t)$ and $y(t)$ respectively. Therefore we have $y_p(t) = h_p(t) \star x_p(t)$, where $\star$ denotes convolution (assume that there is no additive noise).

a) **Show that**

$$y(t) = \sum_{i=1}^{2} x(t - \tau_i)\, e^{-j2\pi f_c \tau_i} \tag{1}$$

Solution: Since $h_p(t) = \sum_{i=1}^{2} \delta(t - \tau_i)$ and $x_p(t) = \text{Re}\big(x(t)e^{j2\pi f_c t}\big)$, we have

$$\begin{aligned}
y_p(t) &= x_p(t) \star \Big(\sum_{i=1}^{2} \delta(t - \tau_i) \Big) \\
&= \sum_{i=1}^{2} x_p(t - \tau_i) \\
&= \sum_{i=1}^{2} \text{Re}\big(x(t - \tau_i) e^{j2\pi f_c (t - \tau_i)} \big) \\
&= \text{Re}\left(\Big[\sum_{i=1}^{2} x(t - \tau_i) e^{-j2\pi f_c \tau_i} \Big] e^{j2\pi f_c t} \right)
\end{aligned} \tag{2}$$

Since $y_p(t) = \text{Re}\big(y(t)e^{j2\pi f_c t}\big)$, comparing this with the expression for $y_p(t)$ in (2) reveals that

$$y(t) = \sum_{i=1}^{2} x(t - \tau_i)\, e^{-j2\pi f_c \tau_i} \tag{3}$$

2) **Linear modulation** (20 points)

Consider linear modulation, where the band-limited complex baseband transmit signal is given by

$$x(t) = \sum_{k=-\infty}^{\infty} x[k]\, p(t - kT). \tag{4}$$

Here $\{x[k]\}$ is the sequence of information bearing symbols, and $p(t)$ is the band-limited pulse shaping filter. Assume that there is no channel filtering, i.e., channel impulse response $h_p(t) = \delta(t)$. At the receiver, the received complex baseband signal $y(t) = x(t) + n(t)$ is passed through a matching filter with impulse response $p(-t)$ (assume $p(\cdot)$ to be a real-valued waveform). The output $z(t) = y(t) \star p(-t)$ is then sampled every T seconds to result in the discrete-time samples $z[k] = z(kT)$. Also, assume that the sequence $\{x[k]\}$ is wide sense stationary and ergodic in autocorrelation.

a) (5 points) **What is the condition on $p(t)$ so that there is no I.S.I. (inter-symbol interference) at the receiver ?**

b) (5 points) **Find the Nyquist pulse having the minimum bandwidth occupancy.**

c) (5 points) **Show that the pulse**

$$s(t) = sinc\Big(\frac{t}{T}\Big) \frac{\cos(\frac{\pi t}{2T})}{1 - (\frac{t}{T})^2} \tag{5}$$

is Nyquist ($sinc(x) \triangleq \sin(\pi x)/(\pi x)$). How much excess bandwidth does $s(t)$ occupy when compared to the minimum bandwidth Nyquist pulse.

d) (5 points) **Under what conditions is the spectral shape of the power spectral density (PSD) of $x(t)$ (in (4)) independent of the inter-symbol period T ?**

Solution:

a) After matched filtering at the receiver, the output is given by

$$\begin{aligned} z(t) &= y(t) \star p(-t) \\ &= \sum_{k=-\infty}^{\infty} x[k]\Big(p(t-kT) \star p(-t)\Big) \; + \; w(t) \\ &= \sum_{k=-\infty}^{\infty} x[k]q(t-kT) \; + \; w(t) \end{aligned} \tag{6}$$

where $q(t) \triangleq p(t) \star p(-t)$, and $w(t) \triangleq n(t) \star p(-t)$. After sampling at every T seconds, the n-th sample is given by

$$\begin{aligned} z[n] &= z(nT) \\ &= \sum_{k=-\infty}^{\infty} x[k]q(nT-kT) \; + \; w(nT) \\ &= x[n]q(0) \; + \; \sum_{k\neq n} x[k]q((n-k)T) \; + \; w(nT). \end{aligned} \tag{7}$$

where $x[n]q(0)$ is the desired term and $\sum_{k\neq n} x[k]q((n-k)T)$ is the inter-symbol interference term. Since, we would like to have no ISI (i.e., $z[n] = x[n]+w(nT)$, it is required that $q(t) = p(t)\star p(-t)$ be such that $q(0) = 1$, and $q(mT) = 0$ for all $m \neq 0$.

b) The Nyquist criterion for no ISI, is equivalently stated in the frequency domain as

$$\frac{1}{T}\sum_{k=-\infty}^{\infty} Q(f+\frac{k}{T}) \;\; = \;\; 1 \tag{8}$$

for all f. Here $Q(f)$ is the Fourier transform of $q(t) = p(t) \star p(-t)$. Infact, $Q(f) = |P(f)|^2$ where $P(f)$ is the Fourier transform of $p(t)$. The frequency domain Nyquist criterion essentially states that, adding together shifted replicas of $Q(f)$ by integer multiples of $1/T$ should result in a constant valued function in the frequency domain. Also $Q(f) = |P(f)|^2$ is band-limited to the same frqeuency range as $P(f)$ (let us say $[-W, W]$).

It can be observed that, if $1/T - W > W$, then there will be no overlap between replicas of $Q(f)$ shifted by consecutive integral multiples of $1/T$, and hence the Nyquist criterion can never be satisfied. Therefore, for the Nyquist criterion to be satisfied, $1/T$ must be less than or equal to $2W$, or in other words the bandwidth occupied must be greater than or equal to $1/T$.

Also, any pulse which satisfies the Nyquist criterion with $1/T - W = W$, must also satisfy $Q(f) = T, |f| \leq W$ and $Q(f) = 0, |f| > W$, since otherwise (8) cannot be satisfied. From here it follows that $p(t) = (1/\sqrt{T})sinc(t/T)$ is the minimum bandwidth occupying Nyquist pulse with bandwidth $1/T$.

c) $s(t)$ is in fact the raised cosine pulse with 50 percent excess bandwidth when compared to the minimum bandwidth Nyquist pulse. It is clear that $s(t = mT) = 1$ only when $m = 0$ and $s(t = mT) = 0$ for all $m \neq 0$.

d) The power spectral density of a linearly modulated complex baseband signal (as given in (4)) is given by

$$S_X(f) \;=\; \frac{1}{T}\sum_{k=-\infty}^{\infty} R_X[k]\, e^{-j2\pi fkT}\; |P(f)|^2 \tag{9}$$

where $P(f)$ is the Fourier transform of $p(t)$ and $R_X[k] = \mathbb{E}[x[m]\, x[m-k]^*]$ is the autocorrelation function of the discrete time sequence $x[k]$. For the spectral distribution of power i.e., the shape of $S_X(f)$ to be independent of T, from (9) it is clear that $\sum_{k=-\infty}^{\infty} R_X[k]\, e^{-j2\pi fkT}$ must be independent of T. Since $R_X[k] = R_X[-k]^*$, we have

$$\sum_{k=-\infty}^{\infty} R_X[k]\, e^{-j2\pi fkT} \;=\; R_X[0] + 2\sum_{k=1}^{\infty} \text{Re}(R_X[k] e^{-j2\pi fkT}). \tag{10}$$

The only way $\sum_{k=1}^{\infty} \text{Re}(R_X[k] e^{-j2\pi fkT})$ is independent of T, is if

$$R_X[k] \;=\; 0 \;\;,\;\; k \neq 0 \tag{11}$$

which means that $\{x[k]\}$ is a sequence of uncorrelated symbols.

3) **(Filtering Channels)** (20 points)

Consider an ISI (filtering) channel with additive noise. The complex baseband discrete-time input and output are related by

$$Y[n] = \sum_{k=0}^{1} h[k]X[n-k] \,+\, W[n] \tag{12}$$

where $Y[n]$ and $X[k]$ are the n-th discrete time output sample and the k-th information symbol respectively. $h[k]\,,\, k = 0, 1$ is the impulse response of the channel filter. $W[n]$ is the additive noise. We will assume that the sequence of random variables $\{W[n]\}$ are independent and identically distributed (i.i.d.). Assume $W[n] = W^I[n] \,+\, jW^Q[n]$ to be complex valued with independent and identically distributed real and imaginary components. Let the real component i.e., $W^I[n]$ have a continuous probability density function (p.d.f.) denoted by $f_W(\cdot)$.

Let the input information symbols belong to some complex-valued alphabet $\mathcal{A} \subset \mathcal{C}$, i.e., $X[k] \in \mathcal{A}$. Also, let the input be time limited, i.e., $X[k] = 0$ for $k < 0$ and $k > 2$. Further, the input information symbols $\{X[k]\}$ are independent of the additive noise.

Give an expression for the maximum likelihood detector given the observed discrete time output samples $y[0], y[1], y[2], y[3]$.

What is maximum likelihood estimate of the transmitted information symbols if we assume that the i.i.d. real and imaginary components of the additive noise are white and Gaussian distributed with zero mean.

Solution: Given the observed discrete time output samples $y[0], y[1], y[2], y[3]$, the maximum likelihood estimate of $(X[0], X[1], X[2])$ is given by

$$(\widehat{X}[0], \widehat{X}[1], \widehat{X}[2]) \;=\; \arg \max_{(x[0],x[1],x[2])\in\mathcal{A}^3} P_{Y\,|\,X}\Big(Y[i] = y[i]\,,\, i = 0, 1, \cdots, 3 \;|\; X[i] = x[i]\,,\, i = 0, 1, 2\Big) \tag{13}$$

where $P_{Y\,|\,X}(\cdot)$ is the joint probability density function of the random output samples conditioned on a given/fixed input sequence of information symbols. Since $Y[n] = \sum_{k=0}^{1} h[k]X[n-k] \,+\, W[n]$, we can simplify the above expression as

$$\begin{aligned}
(\widehat{X}[0], \widehat{X}[1], \widehat{X}[2]) \;&=\; \arg \max_{(x[0],x[1],x[2])\in\mathcal{A}^3} P_{W\,|\,X}\Big(W[i] = (y[i] - \sum_{k=0}^{1} h[k]X[i-k])\,,\, i = 0, 1, \cdots, 3 \\
&\qquad\qquad | \, X[i] = x[i]\,,\, i = 0, 1, 2\Big) \\
&=\; \arg \max_{(x[0],x[1],x[2])\in\mathcal{A}^3} P_{W\,|\,X}\Big(W[i] = (y[i] - \sum_{k=0}^{1} h[k]x[i-k])\,,\, i = 0, 1, \cdots, 3 \\
&\qquad\qquad | \, X[i] = x[i]\,,\, i = 0, 1, 2\Big) \\
&=\; \arg \max_{(x[0],x[1],x[2])\in\mathcal{A}^3} P_{W}\Big(W[i] = (y[i] - \sum_{k=0}^{1} h[k]x[i-k])\,,\, i = 0, 1, \cdots, 3\Big)
\end{aligned} \tag{14}$$

where $P_{W \mid X}(\cdot)$ is the joint pdf of the noise samples conditioned on a given input sequence of information symbols. Note that the last equality in (14) follows from the fact that the additive noise is independent of the input information symbols. Further, since the sequence of noise random variables is independent and identically distributed we have

$$(\widehat{X}[0], \widehat{X}[1], \widehat{X}[2]) = \underset{(x[0],x[1],x[2]) \in \mathcal{A}^3}{\arg\max} \prod_{i=0}^{3} f_W\Big(\mathrm{Re}(y[i] - \sum_{k=0}^{1} h[k]x[i-k])\Big)\, f_W\Big(\mathrm{Im}(y[i] - \sum_{k=0}^{1} h[k]x[i-k])\Big) \tag{15}$$

where we have also used the fact that the real and imaginary components of $W[n]$ are i.i.d.
When the additive noise is white and Gaussian distributed with zero mean and variance σ^2 per real/imaginary component, we have

$$f_W(w) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{w^2}{2\sigma^2}} \tag{16}$$

using which the maximum likelihood detector with AWGN is given by

$$\begin{aligned}(\widehat{X}[0], \widehat{X}[1], \widehat{X}[2]) &= \underset{(x[0],x[1],x[2]) \in \mathcal{A}^3}{\arg\max} \prod_{i=0}^{3} e^{-\frac{\left|y[i] - \sum_{k=0}^{1} h[k]x[i-k]\right|^2}{2\sigma^2}} \\ &= \underset{(x[0],x[1],x[2]) \in \mathcal{A}^3}{\arg\min} \sum_{i=0}^{3} \left|y[i] - \sum_{k=0}^{1} h[k]x[i-k]\right|^2. \end{aligned} \tag{17}$$

4) **(Estimating the unknown phase of a sinusoid of known frequency in Gaussian noise)** (15 points)

Consider, that the received passband signal is given by

$$Y_p(t) \;=\; \cos(2\pi f_c t + \Theta) \;+\; N_p(t) \quad , \quad t \in [0\,,\,T_0] \tag{18}$$

where Θ is the unknown phase and $N_p(t)$ is the AWGN. Given the received waveform $y_p(t)\,,\,t \in [0\,,\,T_0]$, the receiver needs to estimate the unkown phase offset Θ.

Write down the log-likelihood function of the received signal conditioned on the phase offset. Using this compute the maximum likelihood estimate of the unknown phase offset.

Assume that $f_c T_0 \gg 1$.

Solution:

Since the noise is white and Gaussian, the log-likelihood function of the received waveform conditioned on the phase offset is given by

$$\log(L_{Y_p|\Theta}(y_p(t)\,|\,\Theta=\theta)) \;=\; -\frac{\int_0^{T_0}(y_p(t)-\cos(2\pi f_c t+\theta))^2\,dt}{2\sigma^2} \tag{19}$$

where σ^2 is the power spectral density of $N_p(t)$. The maximum likelihood estimate of the unknown phase Θ is given by

$$\begin{aligned}
\Theta_{\text{ML}} &= \arg\max_{\theta\in[-\pi\,,\,+\pi)} L_{Y_p|\Theta}(y_p(t)\,|\,\Theta=\theta) \\
&= \arg\min_{\theta\in[-\pi\,,\,+\pi)} \int_0^{T_0} (y_p(t)-\cos(2\pi f_c t+\theta))^2\,dt \\
&= \arg\max_{\theta\in[-\pi\,,\,+\pi)} \int_0^{T_0} y_p(t)\cos(2\pi f_c t+\theta)\,dt \\
&= \arg\max_{\theta\in[-\pi\,,\,+\pi)} y_I\cos(\theta) \;-\; y_Q\sin(\theta)
\end{aligned} \tag{20}$$

where $y_I \triangleq \int_0^{T_0} y_p(t)\cos(2\pi f_c t)\,dt$ and $y_Q \triangleq \int_0^{T_0} y_p(t)\sin(2\pi f_c t)\,dt$. It then follows that

$$\begin{aligned}
\Theta_{\text{ML}} &= \arg\max_{\theta\in[-\pi\,,\,+\pi)} \sqrt{y_I^2+y_q^2}\,\cos\left[\theta+\tan^{-1}\left(\frac{y_Q}{y_I}\right)\right] \\
&= -\tan^{-1}\left(\frac{y_Q}{y_I}\right)
\end{aligned} \tag{21}$$

5) **(Channel Coding)** (30 points)

Consider a rate $R = 2/3$ binary convolutional encoder with generator matrix

$$G(D) = \begin{bmatrix} 1+D & D & 1+D \\ 1 & 1 & D \end{bmatrix}$$

The D-transforms of the output and the input sequences are related by the generator matrix as follows.

$$\begin{bmatrix} v_1(D) \ v_2(D) \ v_3(D) \end{bmatrix} \ = \ \begin{bmatrix} u_1(D) \ u_2(D) \end{bmatrix} G(D)$$

where $u_1(D) = \sum_n u_1(n)D^n$ and $u_2(D) = \sum_n u_2(n)D^n$ are the D-transforms of the two input sequences $\{u_1(n)\}$ and $\{u_2(n)\}$. Similarly, $v_1(D)$, $v_2(D)$ and $v_3(D)$ are the D-transforms of the three output sequences $\{v_1(n)\}$, $\{v_2(n)\}$ and $\{v_3(n)\}$ respectively.

Assume that the convolutional encoder starts in the all-zero state. Each coded output bit is mapped to a BPSK symbol ($'0'$ is mapped to $+1$, and $'1'$ is mapped to -1). Let $S[k]$ denote the k-th BPSK symbol at the output of the modulation mapper. More precisely, $S[3n] = (-1)^{v_1(n)}$, $S[3n+1] = (-1)^{v_2(n)}$ and $S[3n+2] = (-1)^{v_3(n)}$.

The BPSK symbols are then linearly-modulated to generate the complex baseband transmit signal, which is given by

$$S(t) = \sum_{k=0}^{\infty} S[k]\text{sinc}(\frac{t}{T} - k).$$

Assuming an AWGN channel (no filtering), the received discrete-time complex baseband symbols are given by

$$Y[k] \ = \ S[k] + W[k] \tag{22}$$

where $W[k] = W^I[k] + jW^Q[k]$ has independent and identically distributed (i.i.d.) Gaussian real and imaginary components (mean zero). Further the sequence of complex Gaussian random variables $\{W[k]\}$ is also i.i.d.

a) **Draw the state diagram for the convolutional encoder above.** (15 points)

b) Consider that the receiver performs hard decision decoding (i.e., hard decisions on the received samples $y[k]$, followed by maximum likelihood detection). Suppose that the transmitter sends six information bits followed by two '0' bits (for trellis termination) (information bits refers to the bits at the input of the convolutional encoder). That is the sequence of inputs to the convolutional encoder is $\left\{(u_1[n], u_2[n])\right\}_{n=0}^{n=3}$, where both the transmitter and receiver know that $(u_1[3], u_2[3]) = (0, 0)$.

   Let the received discrete-time sequence be $y[0] = -2.1 + j1.7$,$y[1] = -3.2 - j0.7$,$y[2] = -1.7 + j1.1$,$y[3] = 3.8 + j2.9$, $y[4] = 0.5 - j1.6$,$y[5] = -0.6 - j2.7$, $y[6] = 0.3 + j3.2$,$y[7] = 1.6 - j0.8$, $y[8] = 1.2 - j1.3$, $y[9] = -1.0 + j0.3$, $y[10] = -0.5 + j2.3$, $y[11] = 0.3 - j0.9$.

   Using the Viterbi algorithm (with hard-decisions as input) find out the transmitted information bit sequence having maximum likelihood. Assume that the receiver knows that the transmitter sends six information bits followed by '00'. (15 points)

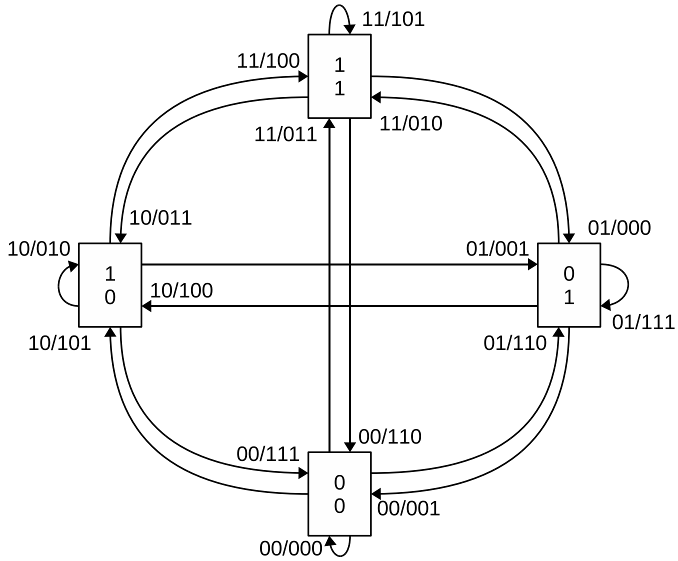

Fig. 1. State Diagram for the given rate $2/3$ convolutional code.

Solution:

a) The state diagram is shown in Fig. 1, where the notation for the state is $(s_1(n)\,,\, s_2(n)) = (u_1(n-1)\,,\, u_2(n-1))$, while the transitions are labeled as $u_1(n)\,u_2(n) \;/\; v_1(n)\,v_2(n)\,v_3(n)$.

b) The survivor paths in the trellis diagram for hard decision Viterbi decoding of the received discrete-time sequence is shown in Fig. 2. Since, the trellis is terminated with a known '00' input at the end, the terminating state in the trellis is '00'.

   Due to hard decision decoding, the metric for a given trellis path is the Hamming distance between the output of the convolutional encoder for this trellis path and the received binary sequence (obtained by making hard decisions on the received complex discrete-time sequence). Since the output of the convolutional encoder is used to modulate only the real component of the information symbols $\{S[k]\}$, and the real and imaginary components of the additive noise are independent, it suffices to only consider the real part of the received discrete-time sequence $\{y[k]\}$.

   In Fig. 2, the accumulated metric for a survivor path is displayed next to it. Also, the maximum likelihood path with the smallest accumulated metric is highlighted in green.

   The maximum likelihood binary sequence at the input of the convolutional encoder is given by $u_1[0] = 1, u_2[0] = 0,\; u_1[1] = 0, u_2[1] = 1,\; u_1[2] = 1, u_2[2] = 1.$

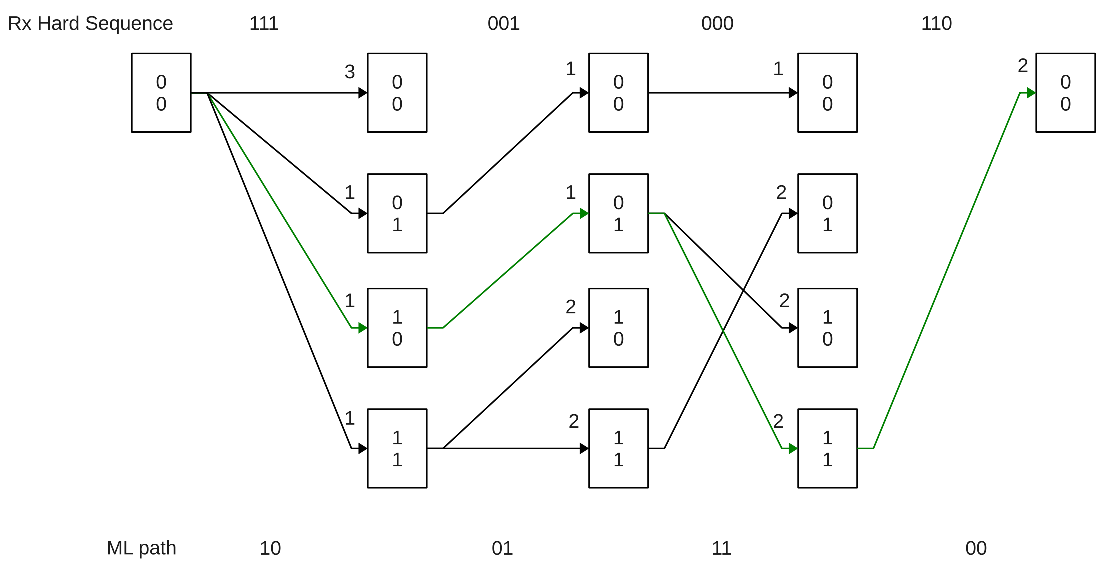

Fig. 2. Hard decision maximum likelihood decoding using the Viterbi algorithm.