Exam in TSKS04 Digital Communication Continuation Course

Exam Code: TEN1

Date & Time: 8:00 - 12:00, 17 August, 2012

Place: TER4

Teacher: Saif Khan Mohammed, tel: 281386

Exam Visit: 08:45 and 11:00

Department: ISY

Allowed aids: Olofsson: Tables and Formulas for Signal Theory

U. Madhow: Fundamentals of Digital Communication

Calculator with clear memory

Grade Translation: Grades 3,4, and 5, are translated to ECTS C,B, and A

Solutions: Within four working days after the exam on the course web page

Grading: Maximum points: 100, Pass: > 41,

Grade 3: $42 - 59$, Grade 4: $60 - 79$, Grade 5: $80 - 100$

Important Instructions: All answers must be given in English

Please write legibly since partial points will be awarded for each question even if the final answer is incorrect

INSTITUTE OF TECHNOLOGY

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1) **Linearly Modulated Signals: Inducing wide sense stationarity using random delay** (30 points)

Consider the linearly modulated signal

$$s(t) \;=\; \sum_{n=-\infty}^{\infty} b[n]p(t-nT). \tag{1}$$

a) (4 points) **Show that s is cyclostationary with respect to the interval T if $\{b[n]\}$ is a stationary discrete-time sequence.**

b) (6 points) **Show that s is wide sense cyclostationary with respect to the interval T if $\{b[n]\}$ is a wide sense stationary (WSS) sequence.**

c) (20 points) Assume that $\{b[n]\}$ is a zero mean wide sense stationary (WSS) sequence with autocorrelation function $R_b[k] = \mathbb{E}\big[b[n]b^*[n-k]\big]$. Let $v(t) = s(t-D)$, where D is a random variable distributed uniformly in $[0\ ,\ T]$ (D is independent of s).

Show that v is wide sense stationary.
(Hint: Show that the mean of $v(t)$ is independent of t. Compute the autocorrelation function of v (i.e., $R_v(t_1,t_2) = \mathbb{E}[v(t_1)v^*(t_2)]$) and show that it depends on t_1 and t_2 only through the time difference t_1-t_2. Be careful to take the expectation over D also, in addition to taking it over $\{b[n]\}$.)

Solution:

a) A random process $s(t)$ is said to be cyclostationary with respect to T, if for any given t, $s(t)$ and $s(t-kT)$ have the same probability distribution for all integer k. Since $s(t)$ is linearly modulated, we have

$$\begin{aligned} s(t-kT) &= \sum_{n=-\infty}^{\infty} b[n]p(t-kT-nT) \\ &= \sum_{n=-\infty}^{\infty} b[n-k]p(t-nT) \end{aligned} \tag{2}$$

The above expression is similar to the expression for $s(t) = \sum_{n=-\infty}^{\infty} b[n]p(t-nT)$, except a shift in the input sequence by k. However, since the sequence $\{b[n]\}$ is stationary, it follows that $s(t)$ and $s(t-kT)$ have the same distribution.

b) A random process $s(t)$ is said to be wide sense cyclostationary with respect to T if for all t, t_1, t_2

$$\begin{aligned} \mathbb{E}[s(t)] &= \mathbb{E}[s(t-T)] \\ \mathbb{E}[s(t_1)s^*(t_2)] &= \mathbb{E}[s(t_1-T)s^*(t_2-T)] \end{aligned} \tag{3}$$

Firstly, using the fact that $s(t)$ is linearly modulated, we have

$$\begin{aligned} \mathbb{E}[s(t-T)] &= \mathbb{E}\Big[\sum_{n=-\infty}^{\infty} b[n]p(t-T-nT)\Big] \\ &= \mathbb{E}\Big[\sum_{n=-\infty}^{\infty} b[n-1]p(t-nT)\Big] \\ &= \sum_{n=-\infty}^{\infty} \mathbb{E}[b[n-1]]p(t-nT) \\ &= \sum_{n=-\infty}^{\infty} \mathbb{E}[b[n]]p(t-nT) \\ &= \mathbb{E}[s(t)] \end{aligned} \tag{4}$$

where the second last equality follows from the fact that $\{b[n]\}$ is a discrete-time wide sense stationary sequence and therefore $\mathbb{E}[b[n]] = \mathbb{E}[b[n-1]]$. This proves the first condition for wide sense cyclostationarity in (3).

Since $\{b[n]\}$ is a discrete-time wide sense stationary sequence, let us denote it's autocorrelation function by $R_b[k] \triangleq \mathbb{E}\Big[b[n]b^*[n-k]\Big]$. For the second condition, we have

$$\begin{aligned} \mathbb{E}[s(t_1-T)s^*(t_2-T)] &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \mathbb{E}\Big[b[n]b^*[m]\Big]p(t_1-T-nT)p^*(t_2-T-mT) \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \mathbb{E}\Big[b[n-1]b^*[m-1]\Big]p(t_1-nT)p^*(t_2-mT) \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_b[n-m]p(t_1-nT)p^*(t_2-mT) \\ &= \mathbb{E}[s(t_1)s^*(t_2)]. \end{aligned} \tag{5}$$

c) This question is problem 2.14(b) of the course text book, and is also included in the set of questions for the first tutorial (http://www.commsys.isy.liu.se/en/student/kurser/TSKS04/AKursprogram/Ilekt). Please refer to the hints provided in the solution document for the first tutorial (the tutorial document on the course web page contains only hints, which should be enough for you to solve the problem). Alternatively, you can also find a similar analysis (not exactly same) in Section A.1.1 of the additional material for lecture 2
(available at http://www.commsys.isy.liu.se/TSKS04/lectures/1/F1-VT2011.pdf).

2) **FSK tone spacing** (20 points)

Consider two real-valued passband pulses of the form

$$s_0(t) = \cos(2\pi f_0 t + \phi_0)\,,\ 0 \le t \le T\,,$$
$$s_1(t) = \cos(2\pi f_1 t + \phi_1)\,,\ 0 \le t \le T\,, \tag{6}$$

where $f_1 > f_0 \gg 1/T$ and $2f_0T\,,\, 2f_1T$ are integers. The pulses are said to be orthogonal if $\int_0^T s_0(t)s_1(t)\,dt = 0$.

a) (10 points) **If $\phi_0 = \phi_1 = 0$, show that the minimum frequency separation such that the pulses are orthogonal is $f_1 - f_0 = \frac{1}{2T}$.**

b) (10 points) **If ϕ_0 and ϕ_1 are arbitrary phases, show that the minimum separation for the pulses to be orthogonal regardless of ϕ_0, ϕ_1 is $f_1 - f_0 = \frac{1}{T}$.**

(Remark: This is the reason why noncoherent FSK requires twice the bandwidth as that required with coherent FSK.)

Solution: This question is problem (2.25) in the course text book and is also included in the third tutorial.

Hints have been provided in the solution document on the tutorial page of the course web page.

3) Maximum Likelihood Sequence Estimation (MLSE) (25 points)

Consider a received signal of the form $y(t) = \sum_l b[l]p(t - lT) + n(t)$, where $b[l] \in \{-1, +1\}$ is the transmitted sequence of information bits, $n(t)$ is AWGN (real), and the pulse shaping waveform $p(t)$ is a real-valued and even function of t (i.e., $p(t) = p^*(t)$, $p(t) = p(-t)$).

At the receiver, the signal $y(t)$ is passed through a receive filter having impulse response $g(t)$ (not necessarily matched to $p(t)$). Suppose that $g(t)$ is such that it satisfies $g(t) \star h(t) = p(t)$ where $h(t) = \delta(t - \frac{T}{4}) + \delta(t + \frac{T}{6})$ (here $\star$ denotes the convolution operation, and $\delta(t)$ denotes the dirac-delta function).

Let $r(t) = y(t) \star g(t)$ denote the output of the receive filter. Let the filtered received signal $r(t)$ be sampled at a rate $1/T_s$ Hz and with a delay of τ seconds, to result in the discrete-time received sequence $r[l] = r(lT_s - \tau)$.

Show that it is possible to implement the MLSE for the original continuous-time signal $y(t)$, using only the discrete-time samples $\{r[l]\}$ for some choice of T_s and τ. Specify a choice of T_s and τ that makes this possible.

Solution:
In general, discrete-time samples of the filtered received signal $r(t)$ can be used to implement MLSE for the original continuous-time signal $y(t)$ only if it is possible to generate the sampled matched filter output from the samples of the filtered received signal.
Let us denote the desired matched-filter output by $z[l] \stackrel{\Delta}{=} z(lT)$, where

$$\begin{aligned} z(t) &\stackrel{\Delta}{=} y(t) \star p^*(-t) \\ &= y(t) \star p(t) \ , \ \text{since } p(t) \text{ is real and even} \\ &= (y(t) \star g(t)) \star h(t) \ , \ \text{since } p(t) = g(t) \star h(t) \\ &= r(t) \star h(t) \ , \ \text{since } r(t) = y(t) \star g(t) \\ &= r(t - \frac{T}{4}) + r(t + \frac{T}{6}) \ , \ \text{since } h(t) = \delta(t - \tfrac{T}{4}) + \delta(t + \tfrac{T}{6}) \end{aligned} \tag{7}$$

Therefore, the desired matched-filter output is

$$z(lT) = r(lT - \frac{T}{4}) + r(lT + \frac{T}{6}) = r\Big((12l - 3)\frac{T}{12}\Big) + r\Big((12l + 2)\frac{T}{12}\Big) \tag{8}$$

By sampling the receive filter output $r(t)$ with a sampling period of $T_s = T/12$ and with a phase shift of $\tau = 0$, the discrete-time filterered received sequence is given by $r[l] \stackrel{\Delta}{=} r(lT_s)$. With these values of $T_s = T/12$ and $\tau = 0$, and using the equation above, we have

$$z[l] = z(lT) = r[12l - 3] + r[12l + 2] \tag{9}$$

which clearly shows that the sampled matched filter output can be constructed from samples of the filtered received signal (sampled at a rate 12 times higher than the signaling rate $1/T$).

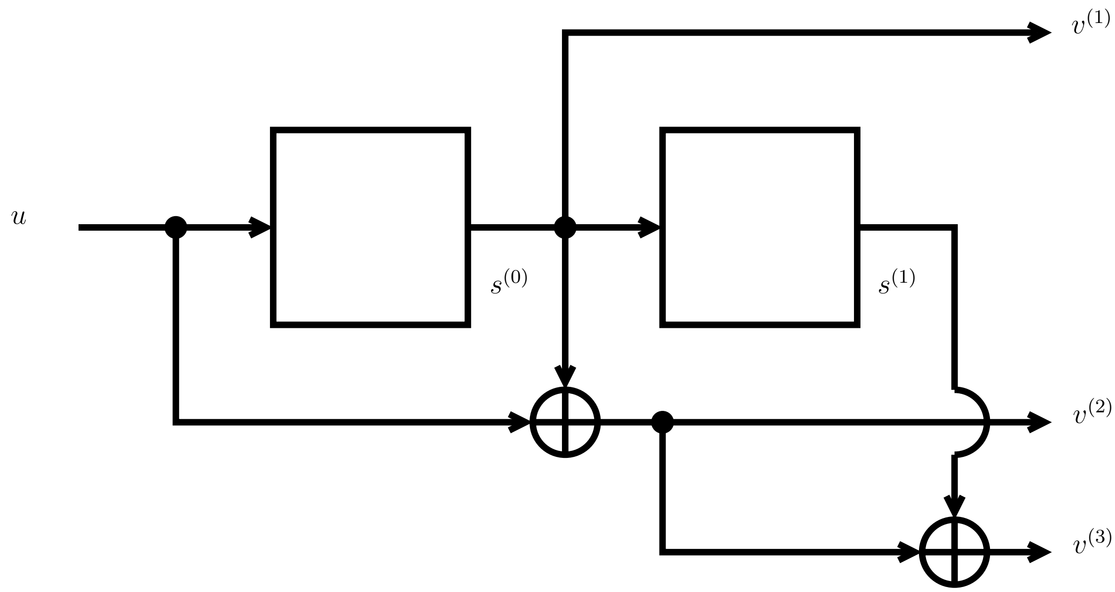

Fig. 1. Rate-1/3 binary convolutional encoder.

4) (25 points) Consider a rate $R = 1/3$ binary convolutional encoder with generator matrix

$$G(D) = \begin{bmatrix} D & 1+D & 1+D+D^2 \end{bmatrix} \tag{10}$$

as shown in Fig. 1.

a) (10 points) **Draw the state diagram for the binary convolutional encoder with generator matrix given in (10).**

b) (15 points) **Draw the signal flow-chart for the convolutional encoder above and compute an expression for the path weight enumerator $T(X) = \sum_i A(i)X^i$ where $A(i)$ is the number of simple-paths (first error event paths in the trellis) whose output sequence has Hamming weight equal to i. Using the expression for $T(X)$, find out the free-distance d_{free} of the above convolutional code, and also, how many different simple-paths exist with Hamming weight equal to 10 ?**

Solution:

a) The state diagram is drawn in Fig. 2.

b) The signal flow chart is drawn in Fig. 3.
From the signal flow chart, the equations relating T_a, T_b and T_c are given by

$$\begin{aligned} T_a &= X^2 + XT_b \\ T_b &= X^3T_a + X^2T_c \\ T_c &= XT_a + X^2T_c. \end{aligned} \tag{11}$$

Finally the path weight enumerator is given by $T(X) = XT_b$. Solving the set of linear equations

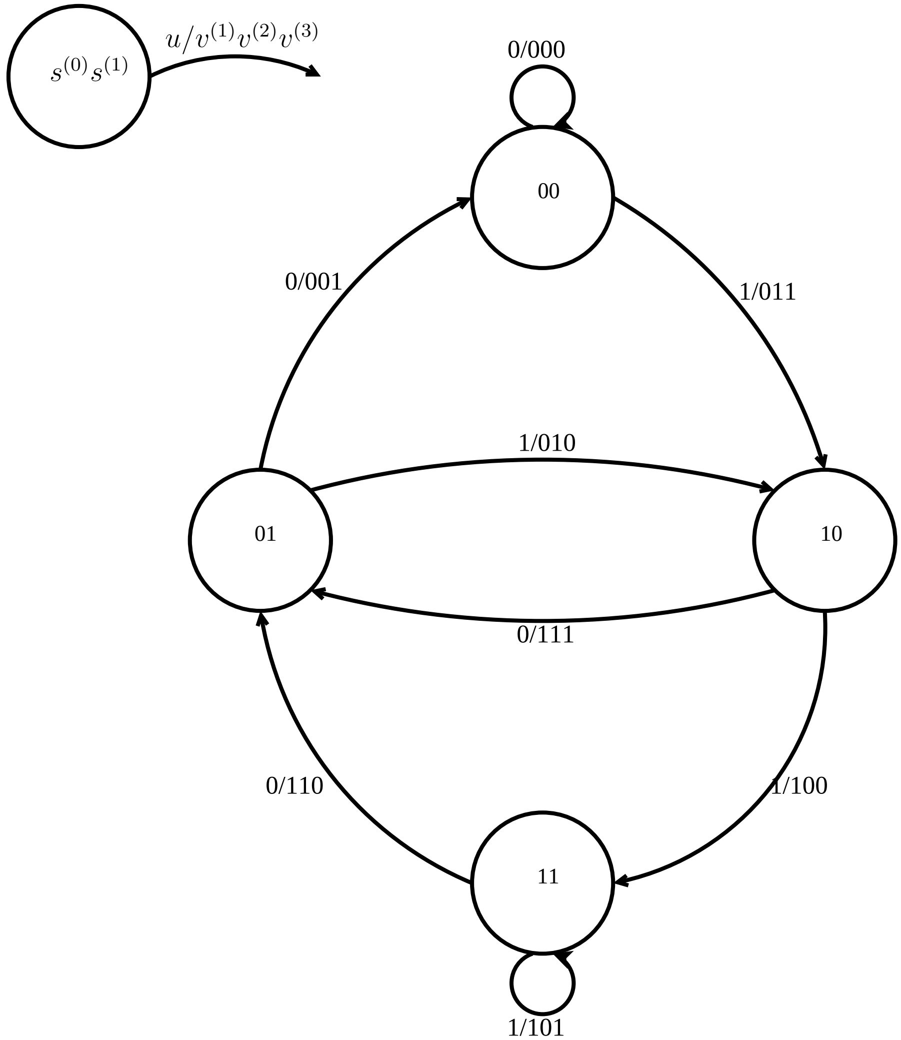

Fig. 2. State Diagram.

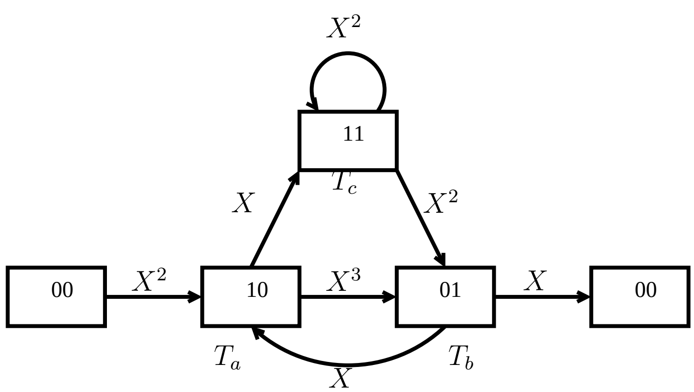

Fig. 3. Signal flow chart.

above we get

$$T(X) = \frac{2X^6 - X^8}{1 - X^2 - 2X^4 + X^6}. \tag{12}$$

On performing long-division, we see that the term with the smallest degree in $T(X)$ is $2X^6$, and therefore $d_{\text{free}} = 6$. Performing long-division further reveals that there are 5 different simple-paths with Hamming weight equal to 10.