TSKS01 Digital Communication

Solutions for the exam 2020-08-22

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. The codewords are (000000), (100110), (010011), (011110), (110101), (111000), (001101), (101011).

b. We have the two signal points

$$\begin{pmatrix} d \\ 0 \end{pmatrix} \text{and} \begin{pmatrix} 0 \\ m \end{pmatrix}.$$

The distance between those points is

$$\sqrt{d^2+m^2}.$$

Finally, the error probability is given by

$$P_{\mathrm{e}} = Q\left(\frac{\sqrt{d^2+m^2}}{\sqrt{2N_0}}\right).$$

The value will depend on the values of d and m.

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2

Each of the five questions can give one point.

- The symbol error probability can be computed exactly when there are only two constellation points, as in BPSK. The expression for M-PSK applies for $M > 2$ and is an approximation based on the nearest neighbor approximation.
- One example is when you have been giving a set of signals and want to find an orthonormal basis for those signals.
- The Hamming coded case can correct errors and is therefore to the left when considering SNRs for which errors are regularly occurring. However, when the SNR is so high that errors are barely occurring, then the uncoded case is more efficient since its coding rate is one, while the coding rate with the Hamming code is smaller than one.

- In hard decoding, each received signal is either correct or in error, and each potentially error is equally likely to happen. In soft decoding, we make use of that each received signal has a different probability of being in error. Those different probabilities are obtained from the ML detection of the transmitted symbols.
- The decision rule is derived under the assumption of Gaussian distributed noise. With this distribution, it is more likely to get small noise realizations (close to zero) than large noise realizations. For this reason it it most likely that the received signal as been obtain by taking the closest constellation point and adding noise to it.

3

a. **False**, the neighbor approximation can both give smaller and larger values. This is why it is called an approximation, not a bound.

b. **True**, the repetition code

c. **True**, in this case, it is more likely to get errors than no errors, so we want to find the codeword that differ the most from the received signal.

d. **False**, these two receivers are identical when it comes to communication performance, but the matched filter receiver might be easier to implement in practice.

e. **True**, the Q-function appears when we compute the probability that the Gaussian distributed noise gives realizations that are larger than the value needed to move the received signal from the constellation point to the decision region of another point.

Problem-solving part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

a. The length n of the code is the number of columns in G, thus $n = 6$. The dimension k of the code is the number of rows in G, thus $k = 3$. The size M of the code is as usual given by $M = 2^k = 8$.

b. We can obtain a generator matrix on systematic form by performing row-operation. For example, we can first add the second row to the first row to obtain:

$$G' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Then we can add the second row of G' to the third row to obtain:

$$G'' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

which is on systematic form.

c. Equality in the Singleton bound means that $n - k = d - 1$. We need to compute the minimum distance to evaluate d the Singleton bound. One way of doing that is to first compute a parity-check matrix based on G'':

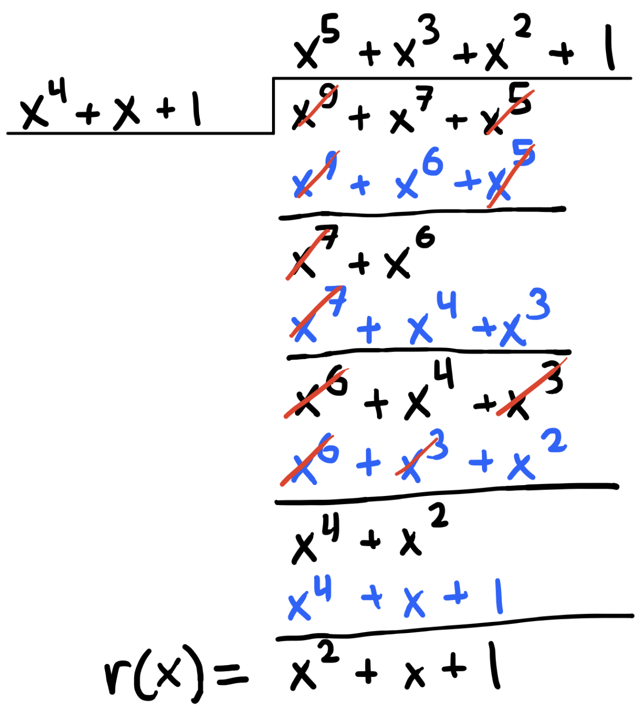
$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Since all the columns are different, the minimum number of linearly dependent columns is 3. Hence, $d = 3$. We have $n - k = 6 - 3 = 3$ and $d - 1 = 2$. We have therefore proved that this code does not satisfy the Singleton bound with equality.

Answer: a. Length: $n = 7$. Dim.: $k = 4$. Size: $M = 16$. **b.** See G'' above. **c.** Singleton bound is not satisfied.

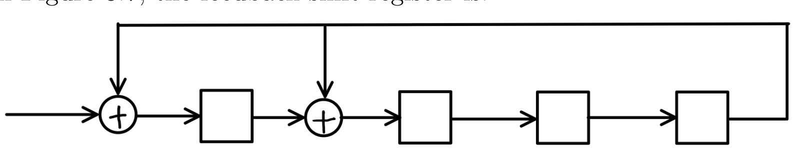
5

a. In this case, $x^{n-k} = x^4$ and, hence, $x^{n-k}m(x) = x^9 + x^7 + x^5$. Next, we should divide $x^{n-k}m(x)$ by the CRC polynomial, which leads to the remainder $r(x) = x^2 + x + 1$. This is the full computation:

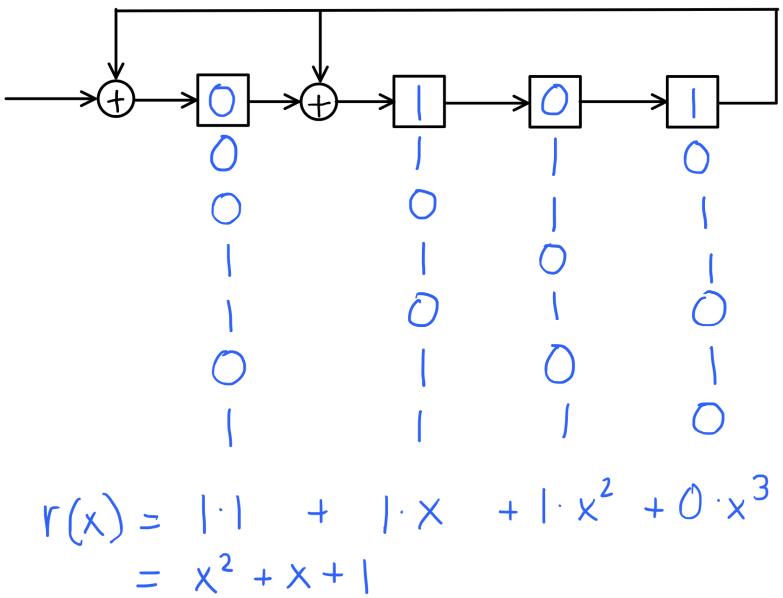


The full codeword is $c(x) = x^9 + x^7 + x^5 + x^2 + x + 1$.

b. Using the structure in Figure 8.7, the feedback shift register is:



c. The shift register is empty in the beginning. We transmit the sequence 1010100000 into the register with the most significant bit first. The feedback loop doesn't make any difference until we have sent 1010 into the register. As we transmit the remaining 100000 into the register, the feedback loop affects the input and the remainder can be computed as follows:



Answer: a. The codeword is $c(x) = x^9 + x^7 + x^5 + x^2 + x + 1$ and $r(x) = x^2 + x + 1$. **b.** See above. **c.** See above.

6
This is a 32-QAM constellation of the kind shown in Figure 6.8.

a. The minimum distance between constellation points is $d_{\min} = \sqrt{2}$. There are 16 points that have 4 nearest neighbors, 8 points that have 3 nearest neighbors, and 8 points that have 2 nearest neighbors. Hence, the average number of nearest neighbors is 3.25 and the approximate symbol error probability is

$$P_e = 3.25Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right) = 3.25Q\left(\frac{1}{\sqrt{N_0}}\right).$$

b. There are points at four different distances from the origin.

- 4 points with distance 1,
- 8 points with distance $\sqrt{5}$,
- 4 points with distance 3,
- 8 points with distance $\sqrt{13}$,
- 8 points with distance $\sqrt{17}$.

This gives the average energy

$$E_{\text{avg}} = \frac{1}{32}(4 \cdot 1 + 8 \cdot 5 + 4 \cdot 9 + 8 \cdot 13 + 8 \cdot 17) = 10.$$

c. Since we need $P_e = 10^{-2} = 3.25Q\left(\frac{1}{\sqrt{N_0}}\right)$, we can use the Q-function table to look for the value a such that $Q(a) = 10^{-2}/3.25 \approx 3.07 \cdot 10^{-3}$. The gives $a = 2.74$ and accordingly

$$N_0 = 1/a^2 \approx 0.13$$

The SNR will then be $E_{\text{avg}}/N_0 \approx 77$.

Answer:

a. $P_e = 3.25Q\left(\frac{1}{\sqrt{N_0}}\right)$.

b. $E_{\text{avg}} = 10$
c. $E_{\text{avg}}/N_0 \approx 77$.

7

a. The distance between the neighboring constellation points is A. The two outmost points have one nearest neighbor and the other three points have two nearest neighbors each. Hence the symbol error probability can be approximated as follows:

$$P_e \approx \frac{1+2+2+2+1}{5} Q\left(\frac{A}{\sqrt{2N_0}}\right) = \frac{8}{5} Q\left(\frac{A}{\sqrt{2N_0}}\right).$$

b. We need to consider all combinations of two points and the distances between them. The points $\pm 2A$ have one neighbor at each of the distances A, $2A$, $3A$, and $4A$. The points $\pm A$ have two neighbors at distance A and one neighbor at each of the distances A, $2A$, and $3A$. The point in the center has two neighbors at distance A and two neighbors at distance $2A$. If we sum up all the possible error events, we get

$$P_e \leq \frac{8}{5} Q\left(\frac{A}{\sqrt{2N_0}}\right) + \frac{6}{5} Q\left(\frac{2A}{\sqrt{2N_0}}\right) + \frac{4}{5} Q\left(\frac{3A}{\sqrt{2N_0}}\right) + \frac{2}{5} Q\left(\frac{4A}{\sqrt{2N_0}}\right)$$

c. The nearest neighbor approximation gives the exact symbol error probability. The reason is that we are already computing the probability that the received signal is anywhere along the line except inside the decision region of the point of interest.

The average energy is $E_{\text{avg}} = 2A^2$ and the maximum energy is $E_{\text{max}} = 4A^2$. Hence, the exact symbol error probability is

$$P_e = \frac{8}{5} Q\left(\frac{A}{\sqrt{2N_0}}\right) = \frac{8}{5} Q\left(\sqrt{\frac{E_{\text{avg}}}{4N_0}}\right) = \frac{8}{5} Q\left(\sqrt{\frac{E_{\text{max}}}{8N_0}}\right).$$

Answer:
a. $P_e \approx \frac{8}{5} Q\left(\frac{A}{\sqrt{2N_0}}\right)$.
b. $P_e \leq \frac{8}{5} Q\left(\frac{A}{\sqrt{2N_0}}\right) + \frac{6}{5} Q\left(\frac{2A}{\sqrt{2N_0}}\right) + \frac{4}{5} Q\left(\frac{3A}{\sqrt{2N_0}}\right) + \frac{2}{5} Q\left(\frac{4A}{\sqrt{2N_0}}\right)$.
c. $P_e = \frac{8}{5} Q\left(\sqrt{\frac{E_{\text{avg}}}{4N_0}}\right) = \frac{8}{5} Q\left(\sqrt{\frac{E_{\text{max}}}{8N_0}}\right)$.