TSKS01 Digital Communication

Solutions for the exam 2020-03-16

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. The codewords are (00000), (01011), (10110), (11101).

b. We have the two signal points

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} \text{and} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The distance between those points is

$$d = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

Finally, the error probability is given by

$$P_\mathrm{e} = Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q(1) \approx 1.5866 \cdot 10^{-1}.$$

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2

Each of the five points can give one point.

- This figure should show four signals that are each phase-shifted by $\pi/2$. An example can be found in the lecture slides.
- The signal space diagram is shown in Figure 6.6(a).
- One example is given in Figure 6.6(a), but there are multiple ways of doing this while satisfying the criterion that adjacent signal points only differ in one bit value.
- An exact expression can be obtained as in Section 5.4.4 with $\alpha = \beta = \sqrt{E_\mathrm{avg}/2}$, leading to $P_e = 2Q\left(\sqrt{\frac{E_\mathrm{avg}}{N_0}}\right) - \left(Q\left(\sqrt{\frac{E_\mathrm{avg}}{N_0}}\right)\right)^2$.

- There are two nearest neighbors for every point at the distance specified at page 123. The approximate symbol error probability is $P_e \approx 2Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right)$.

3

a. **False**, it is the minimum Hamming weight, not the largest.

b. **False**, the signals are orthogonal if $\int_{-\infty}^{\infty} a(t)b(t)dt = 0$.

c. **False**, the sinc pulse has the minimum bandwidth. Any other pulse function requires more bandwidth.

d. **True**, the soft decoding is achieving this by utilizing how likely each of the symbols is to be correct.

e. **True**, the error events that are counted twice happen very seldom in this case.

Problem-solving part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

a. The length n of the code is the number of columns in G, thus $n = 7$. The dimension k of the code is the number of rows in G, thus $k = 4$. The size M of the code is as usual given by $M = 2^k = 16$. There are many different ways to determine the minimum distance d. If we compute the parity-check matrix

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

we notice that the minimum number of linearly dependent columns is 3. Hence, $d = 3$.

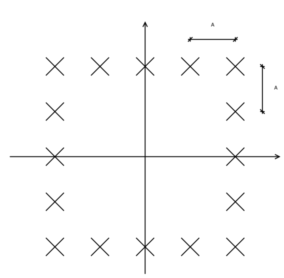
b. With $r = (0111010)$, the syndrome is $s^T = Hr^T = (110)^T$. This is equal to the third column in the parity-check matrix, thus ML decoding gives $C = (0101010)$.

c. We have $2^n = 2^7$ and $2^k \sum_{i=0}^{1} \binom{n}{i} = 2^4(1+7) = 2^7$. Since these terms are equal, the Hamming bound is satisfied.

Answer: a. Length: $n = 7$. Dim.: $k = 4$. Size: $M = 16$. Min. dist.: $d = 3$.
b. The codeword is (0101010).
c. Hamming bound is satisfied.

5

This problem considers the following constellation:



a. There are points at three different distances from the origin.

- 4 points with distance $2A$,
- 8 points with distance $\sqrt{(2A)^2 + A^2} = A\sqrt{5}$,
- 4 points with distance $\sqrt{(2A)^2 + (2A)^2} = 2A\sqrt{2}$.

This gives the average energy

$$E_{\text{avg}} = \frac{1}{16}\left(4(2A)^2 + 8(A\sqrt{5})^2 + 4(2A\sqrt{2})^2\right) = A^2 + \frac{5}{2}A^2 + 2A^2 = \frac{11}{2}A^2.$$

b. The minimum distance between constellation points is $d_{\min} = A$. Each point has two neighbors at this distance. The nearest neighbor approximation then gives

$$P_e \approx \frac{1}{16}\sum_{i=1}^{16} 2Q\left(\frac{A}{\sqrt{2N_0}}\right) = 2Q\left(\frac{A}{\sqrt{2N_0}}\right).$$

c. The symbol error probability with 16-PSK is approximated as

$$P_e \approx 2Q\left(\sqrt{\frac{2E_{\text{avg}}}{N_0}}\sin\left(\frac{\pi}{16}\right)\right) = 2Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\sqrt{2}\sin\left(\frac{\pi}{16}\right)\right) = 2Q\left(0.276\sqrt{\frac{E_{\text{avg}}}{N_0}}\right).$$

For the proposed constellation in the figure, the symbol error probability with $A = \sqrt{\frac{2}{11}E_{\text{avg}}}$ is

$$P_e \approx 2Q\left(\sqrt{\frac{E_{\text{avg}}}{11N_0}}\right) = 2Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\frac{1}{\sqrt{11}}\right) = 2Q\left(0.302\sqrt{\frac{E_{\text{avg}}}{N_0}}\right).$$

The Q-function is a decreasing function. The approximated symbol error probability with 16-PSK is thus higher, which means that the new proposed constellation is the preferred choice.

Answer:

a. $E_{\text{avg}} = \frac{11}{2}A^2$.

b. $P_e \approx 2Q\left(\frac{A}{\sqrt{2N_0}}\right)$.

c. 16-PSK has a higher symbol error probability.

6

The throughput is defined as

$$\frac{\log_2(M)}{T}\frac{k}{n}(1 - P_{\text{packet}}).$$

a. At high SNR, the throughput with uncoded transmission converges to $\frac{\log_2(M)}{T}$, which is $4 \cdot 10^7$ for Technique 5. This implies $\log_2(M) = 4$ and $M = 16$. Hence, 16-QAM could have been used.

b. The generator matrix has $k = 5$ and $n = 10$, thus the coding rate is $k/n = 0.5$. The throughput at high SNR should be $0.5 \cdot 40 = 20$ Mbit/s, which is the case of Technique 3.

c. The coding rate for Technique 1 is 1/8. For a repetition code with $k = 1$, we need $n = 8$. A generator matrix is

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Answer:

a. 16-QAM

b. Technique 3.

c. $n = 8$ and $G = (11111111)$.

7

a. Any Hamming code has a minimum distance of $d = 3$ and can therefore correct 1 error.

b. A codeword is incorrectly detected if there is more than one error. That happens with probability $1 - (1-p)^7 - 7p(1-p)^6$.

c. The erasure channel is deleting coefficients in the codewords, without causing any bit errors. Let c be the number of erasures that occurs in a transmission, then the channel is effectively reducing the code length from n to $n - c$. Note that we need $n - c \geq k$ to have a chance to decode the transmission correctly, thus only $c \leq 3$ is possible.

Since the minimum distance is $d = 3$, we can delete $d - 1 = 2$ coefficients in the codewords and still have distinctly different codewords. This is similar to the principle used in the Singleton bound.

In case of three erasures, we can correct the erasures if the corresponding columns of the parity check matrix are linearly independent. There are 28 such combinations that can occur, while there are 7 combinations that are linearly dependent.

d. A codeword is incorrectly detected with probability $1 - (1-p)^7 - 7p(1-p)^6 - 21p^2(1-p)^5 - 28p^3(1-p)^4$.

Answer: a. 1 error can be corrected.
b. Probability of error is $1 - (1-p)^7 - 7p(1-p)^6$.
c. 2 erasures can always be corrected, there is a 80% chance that 3 erasures can be corrected.
d. Probability of error is $1 - (1-p)^7 - 7p(1-p)^6 - 21p^2(1-p)^5 - 28p^3(1-p)^4$.