TSKS01 Digital Communication

Solutions for the exam 2020-01-16

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. The distance between the two signal points is $d = \sqrt{3^2 + 4^2} = 5$ and then the error probability is

$$P_\mathrm{e} = Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\left(\frac{5}{\sqrt{12}}\right) \approx Q(1.44) \approx 7.4934 \cdot 10^{-2}.$$

b. The syndrome is (0 0 1 1) and since it is non-zero, x is not a codeword.

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2

The answer should give a reasonable description of convolutional codes. Here are some main points that give one point each:

- Description of how to generate codewords.
- Description of how to decode received signals.
- Example of a code with rate 1/3.
- Determining output of the example code for the given input.
- Determining $G(D)$ for the example code.

3

a. **False**, one should only consider the smallest Hamming weight among the non-zero codewords.

b. **False**, it is computed in an entirely different way.

c. **True**, the constellation points lie on a circle in two dimensions.

d. **False**, it is an exact implementation of sequential ML detection.

e. **True**, since $0 \leq Q(x) \leq 1$.

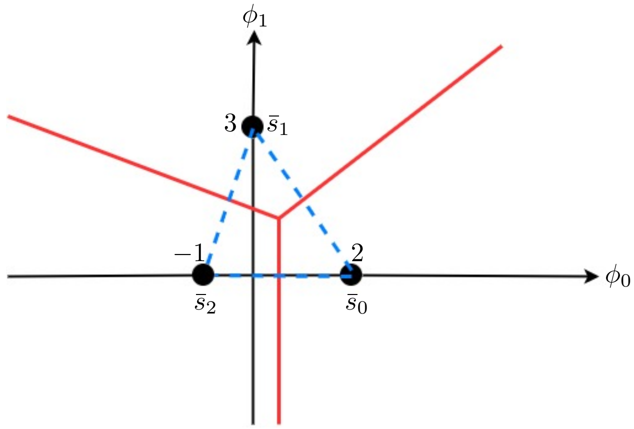

Figure 1: Constellation diagram and decision boundaries for Task 5.

Problem-solving part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

a. We have $n-k=8$ and $k-1=3$, from which we conclude that the dimension is $k=4$ and the length is $n=12$. Hence, the rate is $R=k/n=1/3$.

b. An error has clearly occurred, but this does not mean that the CRC code will detect it. An error is detected whenever the error pattern $w(x)$ divided by the CRC polynomial has a non-zero remainder. Direct computation shows that the remainder is zero and, therefore, the code cannot detect the error. For example, $w(x)$ might coincide with another codeword.

Answer: a. $k=4, n=12, R=1/3$
b. No.

5

a. We have $s_2(t) = \frac{-s_0(t)}{2}$ and the inner product $(s_1, s_0) = 0$. Hence, one selection for the ON basis can be

$$\phi_0(t) = \frac{s_0(t)}{\|s_0\|} = \frac{s_0(t)}{2}, \quad \phi_1(t) = \frac{s_1(t)}{\|s_1\|} = \frac{s_1(t)}{3}.$$

Using this ON basis, we can express the signals as $s_0(t) = 2\phi_0(t)$, $s_1(t) = 3\phi_1(t)$, and $s_2(t) = -\phi_0(t)$. The corresponding vector representations of these signal become

$$\bar{s}_0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \bar{s}_1 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \quad \bar{s}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The constellation diagram is shown as in Fig. 1.

b. The average symbol energy, E_{avg} and the maximum signal energy, $E_{\max}$, are given by

$$E_{\text{avg}} = \frac{2^2 + 3^2 + (-1)^2}{3} = \frac{14}{3}, \quad E_{\max} = 3^2 = 9.$$

c. The decision boundaries are shown with red lines in Fig. 1. The decision region for each symbol is the region between these lines.

d. The distances between different symbols are given by $d_{0,1} = \sqrt{13}$, $d_{0,2} = 3$, and $d_{1,2} = \sqrt{10}$. Hence, the union bound on the symbol error probability is given by

$$P_{\text{e}} \leq \frac{1}{3}\sum_{i=0}^{2}\sum_{j\neq i} Q\left(\frac{d_{i,j}}{\sqrt{2N_0}}\right) = \frac{2}{3}\left(Q\left(\frac{\sqrt{13}}{\sqrt{2N_0}}\right) + Q\left(\frac{3}{\sqrt{2N_0}}\right) + Q\left(\frac{\sqrt{10}}{\sqrt{2N_0}}\right)\right).$$

Answer: a. $\left\{\phi_0(t) = \frac{s_0(t)}{2}, \phi_1(t) = \frac{s_1(t)}{3}\right\}$, $\bar{s}_0 = \begin{pmatrix}2\\0\end{pmatrix}$, $\bar{s}_1 = \begin{pmatrix}0\\3\end{pmatrix}$, $\bar{s}_2 = \begin{pmatrix}-1\\0\end{pmatrix}$. **b.** $E_{\text{avg}} = \frac{14}{3}$, $E_{\max} = 9$. **c.** See Fig. 1. **d.** $P_{\text{e}} \leq \frac{2}{3}\left(Q\left(\frac{\sqrt{13}}{\sqrt{2N_0}}\right) + Q\left(\frac{3}{\sqrt{2N_0}}\right) Q\left(\frac{\sqrt{10}}{\sqrt{2N_0}}\right)\right)$.

6

a. The general ML decision rule is

$$\hat{a} = \underset{a_n\in A}{\text{argmax}}\, f_{\overline{Z}|A}(\overline{z}|a_n) = \prod_{k=0}^{V-1} f_{Z[k]|A}(z[k]|a_n).$$

In this case we have

$$f_{Z[k]|A}(z[k]|a_n) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{\left(z[k]-m_w-b(s_n[k],\ldots,s_n[k-L+1])\right)^2}{2\sigma^2}}, \qquad k \in \{0, 1, \ldots, V-1\}.$$

Hence, the ML decision rule can be expressed as: $\hat{a} = a_i$ where $i = \underset{0\leq n<M^V}{\text{argmin}} \sum_{k=0}^{V-1} \Lambda_k(s_n[k], \mathcal{S}_n[k])$ and $\Lambda_k(s_n[k], \mathcal{S}_n[k]) = \left(z[k] - m_w - b(s[k], \ldots, s[k-L+1])\right)^2$.

b. We should minimize the distance $d(\overline{z} - \overline{m}, \overline{b}_n)$ where $\overline{m}$ is a vector where all the elements are m_w.

c. Yes, the Viterbi algorithm can be used since $\Lambda_k(s_n[k], \mathcal{S}_n[k])$ only depends on $s_n[k]$ and $\mathcal{S}_n[k] = \{s_n[k-1], \ldots, s_n[k-L+1]\}$.

Answer:

7

a. For the constellation 1, the outer signal points on the horizontal axis have the maximum energy and it is

$$E_{\max}^{(1)} = \left(\frac{3d}{\sqrt{2}}\right)^2 = \frac{9}{2}d^2.$$

For the constellation 2, the signal points at $\left(\pm\sqrt{2}d, \pm\frac{d}{\sqrt{2}}\right)$ have the maximum symbol energy and it is

$$E_{\max}^{(2)} = \left(\sqrt{2}d\right)^2 + \left(\frac{d}{\sqrt{2}}\right)^2 = \frac{5}{2}d^2.$$

b. For both constellations, the minimum distance is d. For the first constellation, each signal has two nearest neighbours. Hence, the nearest neighbour (NN) approximation of the symbol error probability is given by

$$P_{\text{e}}^{(NN),1} = 2Q\left(\frac{d}{\sqrt{2N_0}}\right).$$

For the second constellation:

- 4 signal points at $\left(\pm\sqrt{2}d, \pm\frac{d}{\sqrt{2}}\right)$ has 1 nearest neighbour.
- 2 signal points at $\left(\pm\frac{d}{\sqrt{2}}, 0\right)$ has 4 nearest neighbours.
- 2 signal points at $\left(0, \pm\frac{d}{\sqrt{2}}\right)$ has 2 nearest neighbours.

Hence, NN approximation of the symbol error probability becomes

$$P_{\mathrm{e}}^{(NN),2} = \frac{4\cdot 1 + 2\cdot 4 + 2\cdot 2}{8} Q\left(\frac{d}{\sqrt{2N_0}}\right) = 2Q\left(\frac{d}{\sqrt{2N_0}}\right).$$

c. Say that the maximum energy for both constellations are set to $E_{\max}^{(1)} = E_{\max}^{(2)} = E$ and let d_1 and d_2 denote the corresponding minimum distances for the first and second constellation, respectively. Then, we have

$$\frac{9}{2}d_1^2 = E \implies d_1 = \sqrt{\frac{2E}{9}}, \quad \frac{5}{2}d_2^2 = E \implies d_2 = \sqrt{\frac{2E}{5}},$$
$$P_{\mathrm{e}}^{(NN),1} = 2Q\left(\sqrt{\frac{E}{9N_0}}\right), \quad P_{\mathrm{e}}^{(NN),2} = 2Q\left(\sqrt{\frac{E}{5N_0}}\right).$$

We conclude that the second constellation is preferable since it has approximately less symbol error probability (due to the $\frac{9}{5}$ times more SNR compared to the first constellation) for the same maximum symbol energy.

Answer: a. $E_{\max}^{(1)} = \frac{9}{2}d^2$, $E_{\max}^{(2)} = \frac{5}{2}d^2$. **b.** $P_{\mathrm{e}}^{(NN),1} = P_{\mathrm{e}}^{(NN),2} = 2Q\left(\frac{d}{\sqrt{2N_0}}\right)$. **c.** Constellation 2 is preferable.