TSKS01 Digital Communication

Solutions for the exam 2019-04-23

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. The generator matrix has 7 columns and, thus, the length of the code is 7.
b. This is on-off keying with $E_{\max} = 1$ and $N_0 = 1$. Hence,

$$P_e = Q\left(\frac{1}{\sqrt{2}}\right) \approx Q(0.71) \approx 2.3885 \cdot 10^{-1}.$$

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2

The answer should give a reasonable description of link adaptation for packet transmission. Here are some main points that give one point each:

- Explanation of the practical need for link adaptation (e.g., due to variations in signal-to-noise ratio in mobile communication).
- Mathematical description of the throughput formula.
- Description of the parameters that affect the throughput (e.g., signal constellation, coding rate, signal-to-noise ratio, etc.).
- A sketch of the throughput with three different combination of modulation and coding, as a function of the signal-to-noise ratio. It can be similar to the graphs in the course material.
- An explanation of how the sketched combinations of modulation/coding can be used for link adaptation.

3

(a) **True**, it is $Q(\sqrt{E_{\text{avg}}/N_0})$ in both cases.
(b) **False**, soft decoding uses additional information in the ML decoding and can never be worse than hard decoding (usually, it gives a lower error probability).
(c) **False**, it is proportional to the *inverse* of the symbol time.
(d) **True**, it is an efficient implementation of ML detection of a sequence.
(e) **False**, the symbol error probability is the same, but the bit error probability can be reduced by mapping bits to symbols such that each symbol error usually only results in one bit error.

Problem-solving part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

(a) The energy of the pulse can be computed as

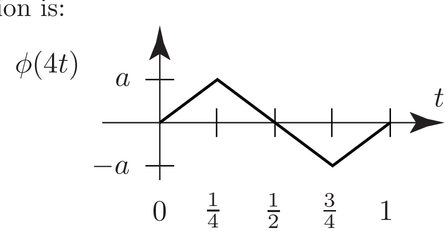
$$E = 4\int_0^1 (ax)^2 dx = \frac{4a^2}{3}.$$

Since BPSK is used, this is both the average and maximum energy. The symbol error probability is

$$P_\mathrm{e} = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{8a^2}{3N_0}}\right).$$

(b) From the table of Q function values, we notice that $Q(3.10) \approx 9.6760 \cdot 10^{-4} \approx 10^{-3}$. (It is also ok to select $Q(3.09) \approx 1.0008 \cdot 10^{-3} \approx 10^{-3}$.) To make $P_\mathrm{e} \approx 10^{-3}$ we thus need $3.10 \approx \sqrt{\frac{8a^2}{3N_0}} = a\sqrt{\frac{4}{3}}$ with $N_0 = 2$, which implies $a \approx 3.10\sqrt{\frac{3}{4}} \approx 2.6847$.

(c) In this case, the new pulse function is:



The signal energy will be

$$E_\mathrm{new} = 4\int_0^{1/4} (4ax)^2 dx = \frac{a^2}{3}.$$

which means that the energy is reduced by a factor 4, because the time duration is reduced by that factor. The new symbol error probability is

$$P_\mathrm{e} = Q\left(\sqrt{\frac{2E_\mathrm{new}}{N_0}}\right) = Q\left(\sqrt{\frac{2a^2}{3N_0}}\right)$$

which is larger than before.

Answer:

(a) $P_\mathrm{e} = Q\left(\sqrt{\frac{8a^2}{3N_0}}\right)$.

(b) $a \approx 2.278$.

(c) The new error probability $P_\mathrm{e} = Q\left(\sqrt{\frac{2a^2}{3N_0}}\right)$ is larger than in a.

5

(a) Based on the description of Hamming codes, the parity check matrix can be expressed as (the columns can be written in any order):

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

(b) Since we wrote the parity check matrix on systematic form we can immediately get the generator matrix:

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(c) There are many ways compute the minimum distance. One way is to consider the smallest number of linearly dependent columns in H. Since all columns are different we need to have $d > 2$. However, since all non-zero binary vectors are columns we can find many combinations of three linearly dependent columns. For example column 1, 2, and 4. This implies that $d = 3$.

(d) The argumentation in c) holds for Hamming codes of arbitrary size. Hence, $d = 3$ for all Hamming codes.

(e) The coding rate is

$$R = \frac{k}{n} = \frac{2^m - m - 1}{2^m - 1} = \frac{1 - \frac{m-1}{2^m}}{1 - \frac{1}{2^m}} \to 1$$

as $m \to \infty$.

Answer:
a. Parity check matrix:

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

b. Generator matrix:

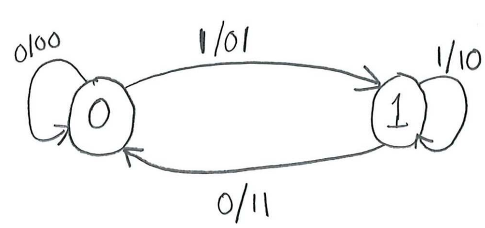
$$G = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

c. $d = 3$.
d. $d = 3$.
e. $R \to 1$.

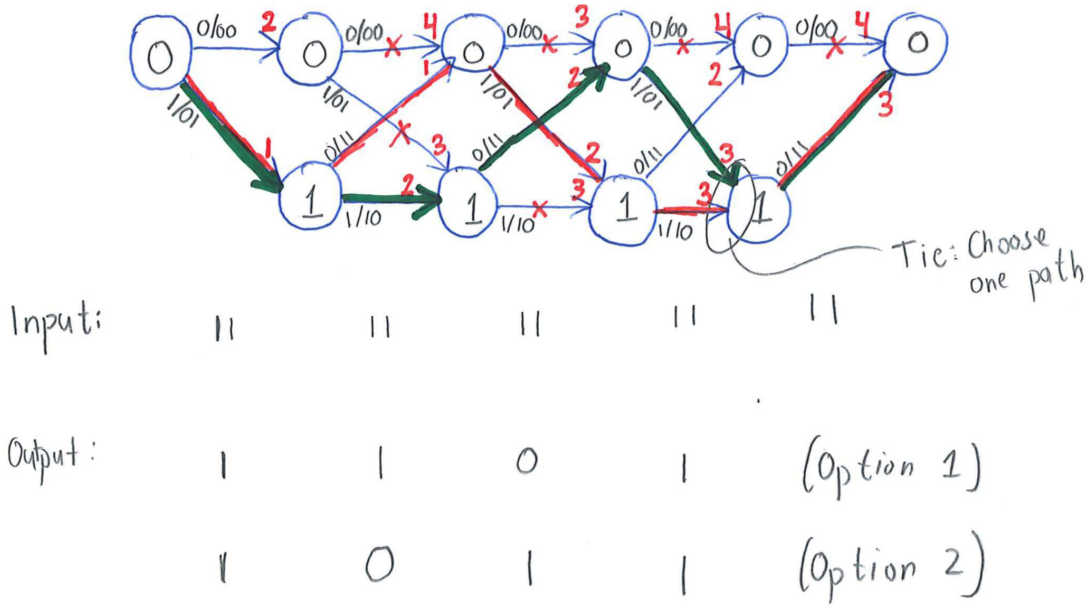
6

a. The D-transforms become $G^{(0)} = D$, $G^{(1)} = 1 + D$.

b. With edges labeled by $a_i/c_i^{(0)}c_i^{(1)}$, we get:



c. The trellis representation and the Viterbi decoding is given below.



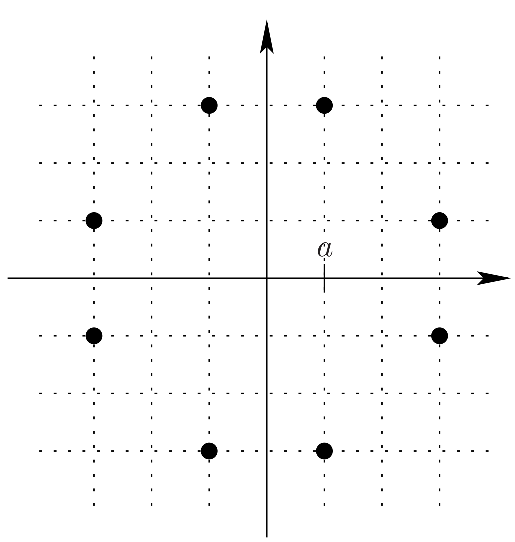
Hence, there are two equally likely signals: 1101 and 1011. These two are equally likely, so one can choose any of them, but the solution must mention this.

Answer: See above.

7

Let us first analyze the two signal constellations, before we solve the two sub-problems. Let a be the length of the side of one small square in the two figures. Furthermore, let E_{max} denote the maximum energy of a signal constellation and let E_{avg} denote the average energy of a signal constellation.

The first signal constellation



The first signal constellation consists of the eight signals

$$(\pm a, \pm 3a), \qquad (\pm 3a, \pm a).$$

Then all signals have energy

$$E = a^2 + (3a)^2 = 10a^2,$$

and consequently we also have

$$E_{\max} = E_{\text{avg}} = 10a^2$$

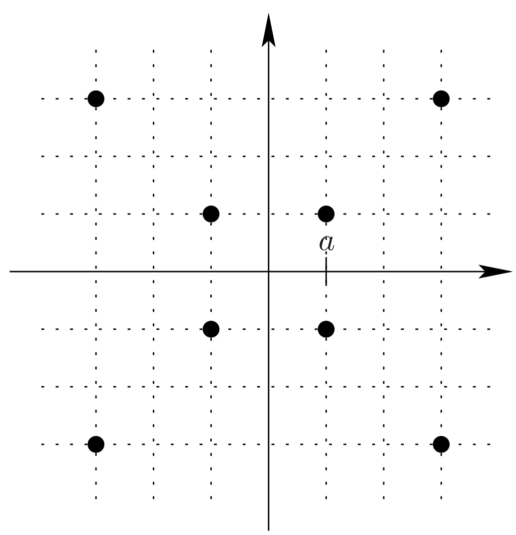
for this signal constellation. The minimum distance d is obviously given by

$$d = 2a,$$

which expressed in terms of our energies is

$$d = \sqrt{\frac{2E_{\max}}{5}} = \sqrt{\frac{2E_{\text{avg}}}{5}}.$$

The second signal constellation



The second signal constellation consists of the eight signals

$$(\pm a, \pm a), \qquad (\pm 3a, \pm 3a),$$

Then the four signals in the middle have energy

$$E_1 = a^2 + a^2 = 2a^2,$$

while the other four signals have energy

$$E_2 = (3a)^2 + (3a)^2 = 18a^2,$$

and consequently we have

$$E_{\max} = 18a^2, \qquad E_{\text{avg}} = 10a^2$$

for this signal constellation. The minimum distance d is again given by

$$d = 2a,$$

which expressed in terms of our energies are

$$d = \sqrt{\frac{2E_{\text{max}}}{9}} = \sqrt{\frac{2E_{\text{avg}}}{5}}.$$

In both cases the average number of nearest neighbours is 1, so the nearest-neighbour approximation of the symbol-error probability is given by

$$P_{\text{e}} \approx Q\left(\frac{d}{\sqrt{2N_0}}\right).$$

Thus, the only thing we need to compare are minimum distances.

(a) For both constellations, the minimum distance is

$$d = \sqrt{\frac{2E_{\text{avg}}}{5}}.$$

and the average number of nearest neighbors is the same. Hence, if the average energy is the same for both constellations, the error probability is the same we may use any of them.

(b) If the maximum energy is the same for both constellations, the situation is different. For the first signal constellation, the minimum distance is given by

$$d = \sqrt{\frac{2E_{\text{max}}}{5}}.$$

expressed in the maximum energy, while the corresponding expression for the second signal constellation is

$$d = \sqrt{\frac{2E_{\text{max}}}{9}}.$$

Thus, using the same maximum energy, we should prefer the first signal constellation, since it gives us the largest minimum distance, and consequently the smallest error probability (since the average number of nearest neighbors is the same).

Answer: a) Equally good; b) First constellation is preferable.