TSKS01 Digital Communication

Solutions for the exam 2019-01-16

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. The minimum distance is 8 and we can therefore correct $\lfloor \frac{8-1}{2} \rfloor = 3$ errors.
b. We have the two signal points at the unit circle, say at angles a and b:

$$\begin{pmatrix} \sin(a) \\ \cos(a) \end{pmatrix} \qquad \begin{pmatrix} \sin(b) \\ \cos(b) \end{pmatrix}$$

The distance between those points is given by

$$d = \sqrt{(\sin(a) - \sin(b))^2 + (\cos(a) - \cos(b))^2}$$

and then the error probability is given by

$$P_e = Q\left(\frac{d}{\sqrt{2N_0}}\right).$$

There are many ways to pick a and b. For example $a = 0$ and $b = \pi$, in which case

$$d = \sqrt{(\sin(0) - \sin(\pi))^2 + (\cos(0) - \cos(\pi))^2} = 2$$

and

$$P_e = Q\left(\frac{2}{\sqrt{4}}\right) = Q(1) \approx 1.5866 \cdot 10^{-1}.$$

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2
Each of the five points can give one point each.

- This figure should show four signals that are each phase-shifted by $\pi/4$. An example can be found in the lecture slides.
- The signal space diagram is shown in Figure 6.6(a).

- One example is given in Figure 6.6(a), but there are multiple ways of doing this while satisfying the criterion that adjacent signal points only differ in one bit value.
- An exact expression can be obtained as in Section 5.4.4 with $\alpha = \beta = \sqrt{E_{\text{avg}}/2}$, leading to $P_e = 2Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right) - \left(Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right)\right)^2$.
- There are two nearest neighbors for every point at the distance specified at page 123. The approximate symbol error probability is $P_e \approx 2Q\left(\sqrt{\frac{E_{\text{avg}}}{N_0}}\right)$.

3

(a) **False**, but it is the only pulse with the minimum bandwidth $1/(2T)$ that satisfies the Nyquist criterion

(b) **True**, syndrome decoding is an implementation of ML detection.

(c) **True**, it is used for timing recovery.

(d) **False**, the states are the values in the delay elements.

(e) **False**. There are many code examples in the book where the all-one codeword does not exist.

Problem-solving part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

(a) We can apply row operations to obtain the equivalent parity-check matrix

$$H_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This matrix is on systematic form and we can therefore obtain the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

(b) The table is:

Information	Codeword
(0 0)	(0 0 0 0 0)
(0 1)	(0 1 1 1 0)
(1 0)	(1 0 1 0 1)
(1 1)	(1 1 0 1 1)

(c) The non-zero codewords have Hamming weights 3, 3, and 4. The minimum distance is therefore 3 and the code can correct one error.

(d) The syndrome is (0 1 1). This can either be explained by errors in the first two positions or in the last two positions. Hence, it is equally likely that either (0 0) or (1 1) was transmitted. The ML principle tells us that we can pick any of these choices.

Answer:

5

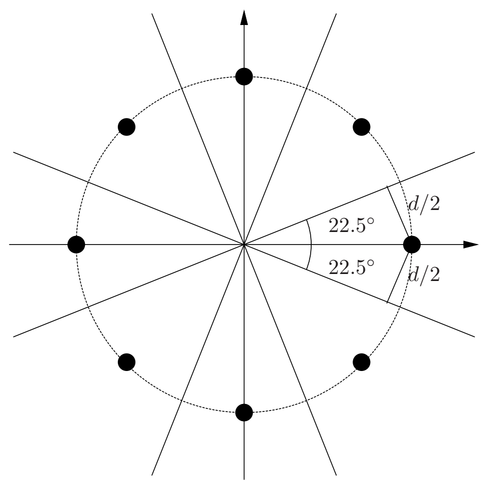
(a) For a CRC code, the message $m(x)$ is a polynomial of degree up to $k-1$. In this case, $k-1=5$ and hence $k=6$.
The CRC polynomial has always the degree $n-k$. Since the degree in this case is $4=n-k$, we get $n=4+k=10$.

(b) The first step in the encoding is to compute $x^{n-k}m(x)$ which in this case is $x^4(x^5+x^2+x)=x^9+x^6+x^5$. Next, we should divide $x^{n-k}m(x)$ by the CRC polynomial, which leads to the remainder $r(x)=x^2$. Finally, the codeword is formed as $x^{n-k}m(x)+r(x)=x^9+x^6+x^5+x^2$.

Answer: a. Length: $n=10$, Dim.: $k=6$.
b. The codeword $c(x)=x^9+x^6+x^5+x^2$.

6

(a) We have 8-PSK.



From the figure, we see that we have

$$\frac{d}{2}=\sqrt{E}\sin(\pi/8)=\sqrt{E}\sin(22.5^\circ),$$

where d is the minimum distance of the signal constellation. Every signal has two nearest neighbours. Therefore, the nearest neighbour approximation gives us the error probability

$$\begin{aligned}P_{\mathrm{e},1} &= 10^{-5} \approx 2Q\left(\frac{a}{\sqrt{N_0/2}}\right)\\ &= 2Q\left(\sqrt{\frac{2E}{N_0}}\sin(22.5^\circ)\right)\end{aligned}$$

for this case. From that we get

$$Q\left(\sqrt{\frac{2E}{N_0}}\sin(22.5^\circ)\right)\approx 5\cdot 10^{-6},$$

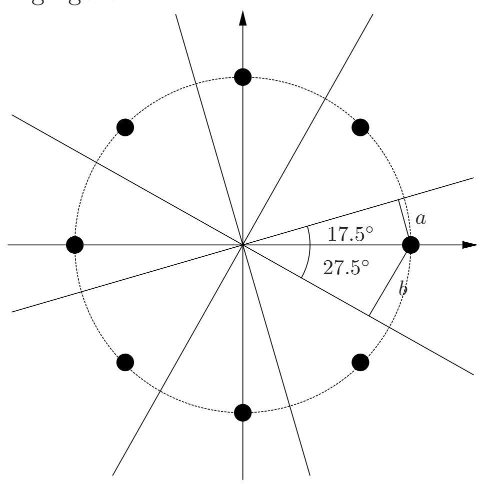
which from the Q-table gives us

$$\sqrt{\frac{2E}{N_0}}\sin(22.5^\circ)\approx 4.42$$

and further

$$\frac{E}{N_0} \approx 66.70.$$

The phase reference of the receiver is assumed to be off by 5°. That means that the decision borders are rotated 5°, and we get the following figure:



From this figure, we get the distances a and b to the decision borders as

$$\begin{aligned} a &= \sqrt{E}\sin(17.5°), \\ b &= \sqrt{E}\sin(27.5°). \end{aligned}$$

This gives us the error probability

$$\begin{aligned} P_{e,2} &\approx Q\left(\frac{a}{\sqrt{N_0/2}}\right) + Q\left(\frac{b}{\sqrt{N_0/2}}\right) \\ &= Q\left(\sqrt{2E/N_0}\sin(17.5°)\right) \\ &\quad + Q\left(\sqrt{2E/N_0}\sin(27.5°)\right) \\ &\approx Q\left(\sqrt{2\cdot 66.70}\sin(17.5°)\right) \\ &\quad + Q\left(\sqrt{2\cdot 66.70}\sin(27.5°)\right) \\ &\approx Q(3.47) + Q(5.33) \approx 2.6\cdot 10^{-4}. \end{aligned}$$

(b) We want to increase the signal energy to a new value E_1, such that the error probability is again 10^{-5}, despite the phase error. We note that the expression of $P_{e,2}$ is dominated completely by the term corresponding to the shorter distance a. This dominance increases if we increase the signal energy. Therefore, we approximate the error probability with this only term, and we get

$$P_{e,1} = 10^{-5} \approx Q\left(\sqrt{\frac{2E_1}{N_0}}\sin(17.5°)\right),$$

which from the Q-table gives us

$$\sqrt{\frac{2E_1}{N_0}}\sin(17.5°) \approx 4.26$$

and further

$$\frac{E_1}{N_0} \approx 100.35.$$

Consider the quotient between the two signal energies

$$\frac{E_1}{E} = \frac{E_1/N_0}{E/N_0} \approx 1.5.$$

Thus, we need to increase the energy by 50%.

Answer:
(a) The error probability becomes $2.6 \cdot 10^{-4}$.
(b) The signal energy has to be increased by 50%.

7

(a) The packet error probability is $1 - \left(1 - Q\left(\sqrt{\frac{2E_\mathrm{b}}{N_0}}\right)\right)^{12}$. This equals 10^{-2} if $E_\mathrm{avg}/N_0 \approx 4.93$. Hence, we need $E_\mathrm{avg}/N_0 \geq 4.93$.
(b) The minimum distance is 7, so the code can correct 3 errors.
(c) We can apply (9.1) for $d = 7$ and $L_p = 23$ to get

$$1 - (1-p)^{23} - 23p\,(1-p)^{22} - 253p^2\,(1-p)^{21} - 1771p^3\,(1-p)^{20}$$

where $p = Q\left(\sqrt{\frac{2E}{N_0}}\right)$.
(d) We need to set $E = E_\mathrm{b}\frac{12}{23}$ to get the correct energy per bit. By evaluating the expression from Part b, we get $1.37 \cdot 10^{-4}$ as packet error probability.

Answer: See above.