TSKS01 Digital Communication

Solutions for the exam 2018-08-25

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. A parity check matrix is an $(n-k) \times n$ matrix where n is the length of the code and k is the dimension. In this case, we have an 8×17 matrix, which means that $n = 17$ and $k = 9$.

b. We have the two signal points

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{och} \quad \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

The distance between these points is

$$d = \sqrt{2^2 + 0^2} = 2.$$

Hence, the error probability is

$$P_{\mathrm{e}} = Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q(\sqrt{5}) \approx 1.2545 \cdot 10^{-2}.$$

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2

Each of the five points can give one point each.

- These are given in Figures 6.2 and 6.6(a).
- These are also found in Figures 6.2 and 6.6(a), but is only important in the case of 4-PSK/QPSK.
- The value is $Q(\sqrt{2E_b/N_0})$ for both modulation types.
- A derivation for BPSK is found in Section 6.2.3. For 4-PSK/QPSK, we should note that the noise is independent between the two dimensions of the received signal. We can use one dimension to decode the first bit and the other dimension to decode the second bit. This reduces the derivations to being the same as in Section 6.2.3.
- It is the same since a 4-PSK/QPSK modulation can be separated into two BPSK modulations, as long as E_b is the same.

3

a. **True**, the constellation points are at the same place in all three cases.

b. **False**, the minimum distance of Hamming codes is always 3.

c. **False**, it is only accurate at high SNR, since then the error probability is dominated by mixing up neighboring constellation points.

d. **True**. This is a consequence of the assumption that every sum of two codewords is another codeword

e. **False**. These are two ways of implementing exactly the same operations in the receiver.

Problem-solving part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

a. The block error probability is $1 - \left(1 - Q\left(\sqrt{\frac{2E_\mathrm{b}}{N_0}}\right)\right)^4$. This equals 10^{-4} if $E_\mathrm{avg}/N_0 \approx 8.224$. Hence, we need $E_\mathrm{avg}/N_0 \geq 8.224$.

b. This expression is given in the equation after (9.2) in the book. In this case we get

$$1 - \left(1 - Q\left(\sqrt{\frac{2E}{N_0}}\right)\right)^7 - 7Q\left(\sqrt{\frac{2E}{N_0}}\right)\left(1 - Q\left(\sqrt{\frac{2E}{N_0}}\right)\right)^6$$

since the Hamming code can correct one error.

c. We need to set $E = E_\mathrm{b}\frac{4}{7}$ to get the correct energy per bit. By evaluating the expression from Part b, we get $2.466 \cdot 10^{-5}$ as packet error probability.

Answer: See above.

5

We are given one codeword (1000101) in a code that is cyclic (which also implies linear) and has dimension four. We therefore want to find four linearly independent codewords to form a generator matrix. There are seven cyclic shifts of that codeword, and all are codewords since the code is cyclic. Let us list those:

(1000101)
(1100010)
(0110001)
(1011000)
(0101100)
(0010110)
(0001011)

There is more than one choice of four linearly independent codewords from that list. We choose the last four and let them be a first generator matrix of the code.

$$G_1 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Next, to get a generator matrix of the code on systematic form, we perform the following row operations on G_1:

- Replace the first row by the sum of the first, third and fourth rows.
- Replace the second row by the sum of the second and fourth rows.

The result is the generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} I_4 & P \end{pmatrix},$$

where we have

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

As usual, we get a parity chack matrix from the systematic generator matrix as

$$H = \begin{pmatrix} P^\mathsf{T} & I_3 \end{pmatrix}, = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

Let $\bar{h}_i$ denote the i-th column of H. We notice that those columns are all non-zero three-dimensional binary vectors, and each of them occur exactly once. Thus, we are dealing with a Hamming-[7,4,3] code, which we know is a perfect code with minimum distance 3 and error correction capability 1. To decode the received vector $\bar{x} = (1010001)$, we calculate the corresponding syndrome:

$$\begin{aligned} \bar{s} = H\bar{x}^\mathsf{T} = \bar{h}_1 + \bar{h}_3 + \bar{h}_7 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \bar{h}_6 \end{aligned}$$

We should therefore flip bit number 6 in $\bar{x}$. The resulting codeword is therfore (1010011), which is the output of the decoder or the corresponding information vector. However, the problem formulation does not reveal what generator matrix is used by the encoder. Therefore, we cannot determine the information vector.

Answer: It will be interpreted as the codeword (1010011).

6

a. The general ML decision rule is

$$\hat{a} = \underset{a_n \in A}{\operatorname{argmax}} f_{\overline{Z}|A}(\overline{z}|a_n) = \prod_{k=0}^{V-1} f_{Z[k]|A}(z[k]|a_n).$$

In this case we have

$$f_{Z[k]|A}(z[k]|a_n) = \frac{1}{\sqrt{\pi N_0}}\, e^{-\frac{\left(z[k]-m_w-b(s_n[k],\ldots,s_n[k-L+1])\right)^2}{N_0}}, \qquad k \in \{0,1,\ldots,V-1\}.$$

Hence, the ML decision rule can be expressed as: $\hat{a} = a_i$ where $i = \underset{0\le n<M^V}{\operatorname{argmin}} \sum_{k=0}^{V-1} \Lambda_k(s_n[k], \mathcal{S}_n[k])$ and $\Lambda_k(s_n[k], \mathcal{S}_n[k]) = \big(z[k] - m_w - b(s[k], \ldots, s[k-L+1])\big)^2$.

b. We should minimize the distance $d(\overline{z} - \overline{m}, \overline{b}_n)$ where $\overline{m}$ is a vector where all the elements are m_w.

c. Yes, the Viterbi algorithm can be used since $\Lambda_k(s_n[k], \mathcal{S}_n[k])$ only depend on $s_n[k]$ and $\mathcal{S}_n[k] = \{s_n[k-1], \ldots, s_n[k-L+1]\}$.

Answer:

7

Obviously, all signals have the same energy, which then is our average energy E. We start by determining E. We have

$$E = ||s_0||^2 = \int_0^T s_0^2(t)\ dt = \frac{1}{3}A^2T.$$

Obviously, $s_0(t)$ and $s_1(t)$ are linearly independent, while $s_2(t)$ and $s_3(t)$ are (trivial) linear combinations of $s_0(t)$ and $s_1(t)$. Thus, the signals span a two-dimensional space. In order to get a vector representation of the signals, we determine α, which we define to be the angle between $s_0(t)$ and $s_1(t)$. It is given by

$$\cos(\alpha) = \frac{(s_0, s_1)}{||s_0|| \cdot ||s_1||} = \frac{(s_0, s_1)}{E}$$

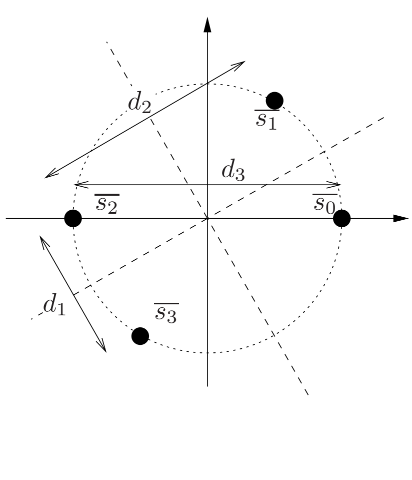
The correlation between $s_0(t)$ and $s_1(t)$ is given by

$$(s_0, s_1) = \int_0^T s_0(t)s_1(t)\ dt = \frac{1}{6}A^2T.$$

Combine those equations and insert E, and we get

$$\cos(\alpha) = \frac{A^2T/6}{A^2T/3} = \frac{1}{2},$$

which finally gives us $\alpha = \pi/3$. This gives us the following signal space diagram, where the dashed lines are borders between the decision regions for ML detection.



The distances d_1, d_2 and d_3 are easily obtained as

$$\begin{aligned} d_1 &= \sqrt{E}, \\ d_2 &= \sqrt{3E}, \\ d_3 &= 2\sqrt{E}, \end{aligned}$$

using geometry. Define q_1, q_2 and q_3 to be the values of the Q-function corresponding to the distances d_1, d_2 and d_3. Then we have

$$\begin{aligned} q_1 &= Q\left(\frac{\sqrt{E}}{\sqrt{2N_0}}\right), \\ q_2 &= Q\left(\frac{\sqrt{3E}}{\sqrt{2N_0}}\right), \\ q_3 &= Q\left(\frac{2\sqrt{E}}{\sqrt{2N_0}}\right). \end{aligned}$$

a. The borders between the decision regions are orthogonal, which makes it possible for us to give an exact expression of the error probability. The error probability is given as

$$P_e = q_1 + q_2 - q_1 q_2.$$

b. Each signal point has one signal on each on distances d_1, d_2 and d_3. Therefore the union bound gives us

$$P_e \leq q_1 + q_2 + q_3.$$

Each signal point has one signal on minimum distance, which is d_1, and we get the nearest-neighbour approximation

$$P_e \approx q_1$$

c. The problem does not state what kind of comparison is to be done. Therefore there are several possibilities. The comparison should, however, be reasonable in the sense that we actually compare soemting interesting, for example one of the following.

- Comparing the power spectral densities of the constellations, and comment about bandwidth. This is hard, since we do not have any Fourier transform table around. I do not expect anyone to actually do this.
- Comparing the error probabilities for the two constellation for the same average signal energy, perhaps for some specific SNR. This is perfectly possible to do.
- Comparing the minimum distances of the two constellations for the same average signal energy. This is of course related to the error probability.
- Comparing the needed average signal energy to achieve the same error probability.

We choose the third alternative. The minimum distance for our constellation is according to the above $\sqrt{E}$. For 4-PSK, the minimum distance is $\sqrt{2E}$. So, in this sense, 4-PSK is a better alternative. To get the same minimum distance for our signal constellation as for 4-PSK, we need twice the average energy. 4-PSK is therefore $10\log_1 0(2) = 3$ dB better than our constellation in terms of minimum distance.

Answer:

a. $P_e = q_1 + q_2 - q_1 q_2$.

b. Union bound: $P_e \leq q_1 + q_2 + q_3$.
Nearest neighbour: $P_e \approx q_1$.

c. 4-PSK is 3 dB better than our constellation in terms of minimum distance.

In the above, we have

$$
\begin{aligned}
q_1 &= Q\left(\frac{\sqrt{E}}{\sqrt{2N_0}}\right), \\
q_2 &= Q\left(\frac{\sqrt{3E}}{\sqrt{2N_0}}\right), \\
q_3 &= Q\left(\frac{2\sqrt{E}}{\sqrt{2N_0}}\right).
\end{aligned}
$$