TSKS01 Digital Communication

Solutions for the exam 2018-04-03

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. We have the two signal points

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The distance between those points is given by

$$d = \sqrt{0^2 + 2^2} = 2.$$

Finally, the error probability is given by

$$P_\mathrm{e} = Q\left(\frac{d}{\sqrt{2N_0}}\right) \approx Q(1.49) \approx 6.81 \cdot 10^{-2}.$$

b. The error detection capability is given by

$$v = d - 1 = 13.$$

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2

The answer should give a reasonable description of the concept of eye patterns, as described in Section 8.2 in the book. Here are some main points that give one point each:

- The eye pattern is generated by producing random sequences of data and sending them over a channel.
- The eye diagrams shows the output of a matched filter at different time instants.
- An illustration of an eye pattern should look like the ones in Figure 8.3, Figure 8.6, Figure 8.7, or Figure 8.11 in the book.
- The reception is good if the “eyes” are widely open, so that the signals are clearly separable at the intended sampling times. Synchronization errors will move the sampling times away from these positions.

- The shape of the basis function determines the shape of the output at the matched filter, as illustrated in Figure 8.1 and Figure 8.5. We want the matched filter output to be relatively flat around the ideal sampling point. This leads to better SNR, which in turn gives fewer errors and better error decoding capability.

3

a. **False**, it is only the non-zero codewords that should be considered.

b. **False**, it is a decreasing function, as can be seen from the table on the last page.

c. **True**, the union bound counts some random events twice, leading to an upper bound on the error probability.

d. **False**, it is used for timing recovery.

e. **True**.

Problem-solving part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

a. Based on the description of Hamming codes, the parity check matrix can be expressed as:

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

b. Since we wrote the parity check matrix on systematic form we can immediately get the generator matrix:

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

c. This table is found on Page 157.

d. The Hamming weights are 0, 4, and 7.

Answer: See above.

5

a. Both constellations have 4 points with energy r_1^2 and 4 points with energy r_2^2, hence

$$E_{avg,1} = E_{avg,2} = \frac{1}{8}\left(4r_1^2 + 4r_2^2\right) = \frac{r_1^2 + r_2^2}{2}.$$

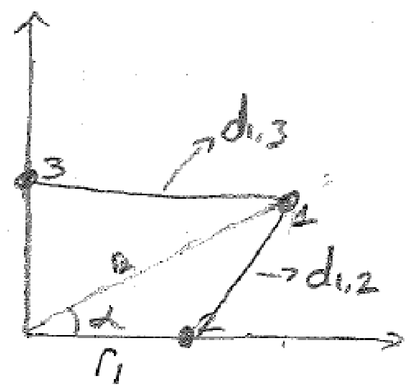
b. There are many ways to compute the probability of error. We use the nearest neighbor approximation

$$P_e \approx \frac{1}{M} \sum_{i=0}^{M-1} n_i Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right).$$

For the first constellation, we have $d_{\min} = r_2 - r_1$ and

$$P_{e,1} \approx \frac{1}{8} \sum_{i=0}^{7} Q\left(\frac{r_2 - r_1}{\sqrt{2N_0}}\right) = Q\left(\frac{1}{2\sqrt{2N_0}}\right).$$

For the second constellation, we focus on one of the constellation points shown in the figure below,



where the distances are given by

$$\begin{aligned} d_{1,2} &= \sqrt{(r_2 \sin\alpha)^2 + (r_2 \cos\alpha - r_1)^2} = \sqrt{\frac{5}{4} - \cos\alpha}\,, \\ d_{1,3} &= \sqrt{(r_2 \cos\alpha)^2 + (r_2 \sin\alpha - r_1)^2} = \sqrt{\frac{5}{4} - \sin\alpha}\,, \end{aligned}$$

and the probability of error can be stated as

$$P_{e,2} \approx \max\left(Q\left(\frac{d_{1,2}}{\sqrt{2N_0}}\right), Q\left(\frac{d_{1,3}}{\sqrt{2N_0}}\right)\right).$$

c. The optimal α can be found by solving the following problem

$$\underset{\alpha}{\operatorname{argmin}}\, P_{e,2} = \underset{\alpha}{\operatorname{argmin}} \max\left(Q\left(\frac{d_{1,2}}{\sqrt{2N_0}}\right), Q\left(\frac{d_{1,3}}{\sqrt{2N_0}}\right)\right) = \underset{\alpha}{\operatorname{argmax}} \min\left(d_{1,2}, d_{1,3}\right).$$

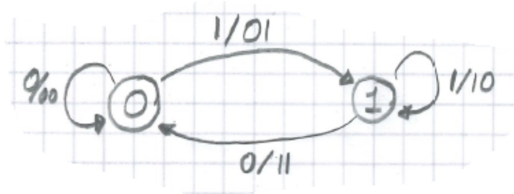
Hence the optimum α must satisfy $d_{1,2} = d_{1,3}$ which is achieved by choosing α as close to 45° as possible.

Answer:

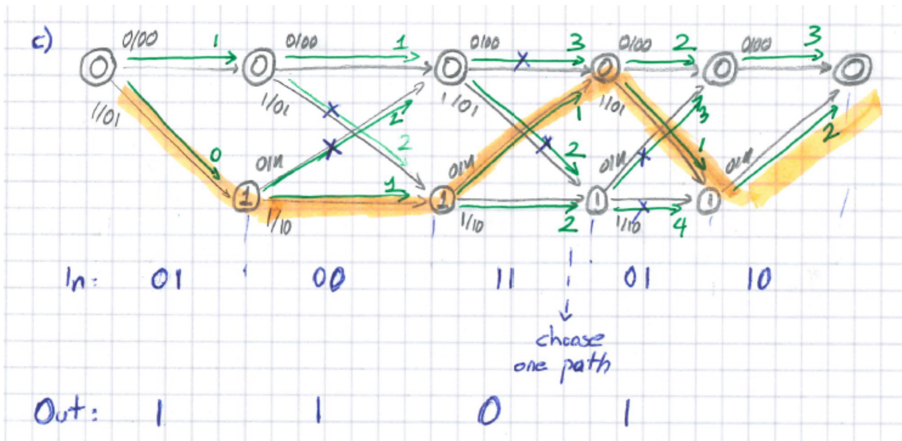
6

a. The D-transforms become $G^{(0)} = D$, $G^{(1)} = 1 + D$.

b. With edges labeled by $a_i/c_i^{(0)}c_i^{(1)}$, we get:



c. The trellis representation and the Viterbi decoding is given below.



Hence, the decoded bits are 1101.

Answer: See above.

7

First consider the uncoded case. Then each sent bit on the channel has energy E_b. Let p denote the error probability of BPSK for that signal energy. Then we have

$$p = Q\left(\sqrt{\frac{2E_\mathrm{b}}{N_0}}\right) = Q(2) \approx 2.2750 \cdot 10^{-2}.$$

The block-error probability P_e for a 4-bit block is then given by

$$\begin{aligned} P_\mathrm{e} &= 1 - \Pr\{\text{all 4 bits received correctly}\} \\ &= 1 - (1-p)^4 \approx 8.7 \cdot 10^{-2}. \end{aligned}$$

Now consider the encoded case. We have the energy $4E_\mathrm{b}$ available for the 7-bit codeword. Hence, each codeword bit is allowed to use $\frac{4}{7}E_\mathrm{b}$ for its modulation. Let q denote the error probability of BPSK for that signal energy. Then we have

$$q = Q\left(\sqrt{\frac{8E_\mathrm{b}}{7N_0}}\right) \approx Q(1.51) \approx 6.5522 \cdot 10^{-2}.$$

The binary Hamming-[7, 4] code is perfect with minimum distance 3. Thus, we can place decoding spheres of radius 1 around each codeword, and those spheres completely fill the vector space. The block-error probability P_e for a 4-bit information block is then given by

$$\begin{aligned} P_\mathrm{e} &= 1 - \Pr\{\text{at least 6 bits of 7 received correctly}\} \\ &= 1 - \Pr\{\text{7 bits correct}\} - \Pr\{\text{6 bits correct}\} \\ &= 1 - (1-q)^7 - 7q(1-q)^6 \approx 7.2 \cdot 10^{-2}. \end{aligned}$$

Answer: Uncoded: $P_{\mathrm{e}} \approx 8.7 \cdot 10^{-2}$ Coded: $P_{\mathrm{e}} \approx 7.2 \cdot 10^{-2}$