TSKS01 Digital Communication

Solutions for the exam 2018-01-11

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. The length of the code is the number of columns in G, i.e. we have $n = 16$. The dimension of the code is the number of rows in G, i.e. we have $k = 5$.

b. We have the two signal points

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{och} \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

The distance between those points is given by

$$d = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

Finally, the error probability is given by

$$P_\mathrm{e} = Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\big(\sqrt{2.5}\big) \approx 5.7 \cdot 10^{-2}.$$

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2

The answer should give a reasonable description of link adaptation for packet transmission. Here are some main points that give one point each:

- Explanation of the practical need for link adaptation (e.g., due to variations in signal-to-noise ratio in mobile communication).
- Mathematical description of the throughput formula.
- Description of the parameters that affect the throughput (e.g., signal constellation, coding rate, signal-to-noise ratio, etc.).

- A sketch of the throughput with four different combination of modulation and coding, as a function of the signal-to-noise ratio. It can be similar to the graphs in the course material.
- An explanation of how the sketched combinations of modulation/coding can be used for link adaptation.

3

a. **False**, only repetition codes of odd length are perfect codes.

b. **True**, the signal has either zero or a non-zero amplitude.

c. **False**, Inter-symbol interference occur when the Nyquist criterion is *not* satisfied.

d. **True**, it is a sequential implementation of the ML detector.

e. **True/false**. It is true that the frequency is the same (and also the phases are the same or strongly related) when using the standard PLL in Figure 10.12. However, this is not the case for Costas loop. Since the question was ambiguous, we gave points also for both True and False.

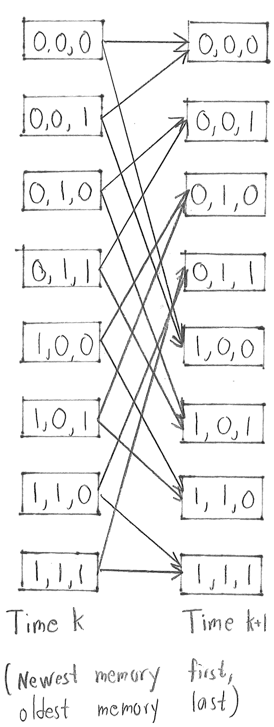
Problem-solving part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

a. A dispersive L-tap channel has a memory of $L-1$ previous received signals. Hence, when using a binary modulation, such as OOK, 2^{L-1} states are required to describe the memory.

b. In this case we need $2^3 = 8$ states. The trellis representation of the state transition between time k and time $k+1$ is:



Answer: a. Memory: $L-1$, Number of states: 2^{L-1} **b.** See state transition above.

5

a. For a CRC code, the message $m(x)$ is a polynomial of degree up to $k-1$. In this case, $k-1=4$ and hence $k=5$.
The CRC polynomial has always the degree $n-k$. Since the degree in this case is $4=n-k$, we get $n=4+k=9$.

b. The first step in the encoding is to compute $x^{n-k}m(x)$ which in this case is x^8+x^5. Next, we should divide $x^{n-k}m(x)$ by the CRC polynomial, which leads to the remainder $r(x)=x^3+x^2+x$.
Finally, the codeword is formed as $x^{n-k}m(x)+r(x)=x^8+x^5+x^3+x^2+x$.

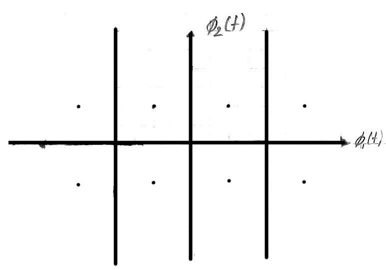
Answer: a. Length: $n=9$, Dim.: $k=5$.
b. The codeword $c(x)=x^8+x^5+x^3+x^2+x$.

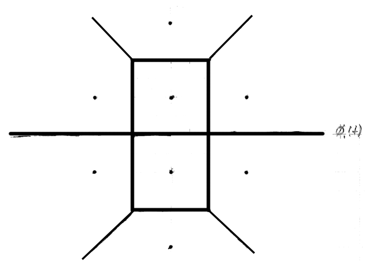
6

a) For the constellation on the left, there are 4 points with energy $\frac{\triangle^2}{2}$ and 4 points with energy $\frac{5\triangle^2}{2}$. Hence, the average energy is $\frac{3\triangle^2}{2}$.

For the constellation on the right, there are 4 points with energy $\frac{5\triangle^2}{4}$, 2 points with energy $\frac{9\triangle^2}{4}$ and 2 points with energy $\frac{\triangle^2}{4}$. Hence, the average energy is $\frac{5\triangle^2}{4}$.

b) The decision regions (based on smallest Euclidean distance) are:





c) There are many possible ways to compute probability of error. Under high SNR assumption nearest neighbor approximation can be utilized as follows

$$P_e \approx \frac{1}{M}\sum_{i=0}^{M-1} n_i Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right).$$

Using this approximation, the probability of error for constellation on the left is

$$P_e \approx \frac{1}{8}\left(4\cdot 2\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)+4\cdot 3\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)\right)=\frac{20}{8}Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right),$$

and similarly for the constellation on the right, we have

$$\begin{aligned} P_e &\approx \frac{1}{8}\left(2\cdot 4\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)+2\cdot 1\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)+4\cdot 2\cdot Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)\right), \\ &= \frac{18}{8}Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right). \end{aligned}$$

Using union bound or computing the exact error probability is also possible.

d) Based on the results obtained in parts **a)** and **c)**, the constellation on the right results in a lower error probability while using less average energy. Hence, we may conclude the constellation on the right is superior. This is not the only way to compare two constellations and other answers based on different criteria are also acceptable.

Answer: See above.

7

a. Any Hamming code has a minimum distance of $d = 3$ and can therefore correct 1 error.

b. A codeword is incorrectly detected if there are 2 bit errors. That happens with probability $1-(1-p)^7-7p(1-p)^6$.

c. The erasure channel is deleting coefficients in the codewords, without causing any bit errors. Let c be the number of erasures that occurs in a transmission, then the channel is effectively reducing the code length from n to $n-c$. Note that we need $n-c \geq k$ to have a chance to decode the transmission correctly, thus only $c \leq 3$ is possible.

Since the minimum distance is $d = 3$, we can delete $d-1 = 2$ coefficients in the codewords and still have distinctly different codewords. This is similar to the principle used in the Singleton bound.

In case of three erasures, we can correct the erasures if the corresponding columns of the parity check matrix are linearly independent. There are 28 such combinations that can occur, while there are 7 combinations that are linearly dependent.

d. A codeword is incorrectly detect with probability $1-(1-p)^7-7p(1-p)^6-21p^2(1-p)^5-28p^3(1-p)^4$.

Answer: **a.** 1 error can be corrected.
b. Probability of error is $1-(1-p)^7-7p(1-p)^6$.
c. 2 erasures can always be corrected, there is a 80% chance that 3 erasures can be corrected.
d. Probability of error is $1-(1-p)^7-7p(1-p)^6-21p^2(1-p)^5-28p^3(1-p)^4$.