TSKS01 Digital Communication

Solutions for the exam 2017-08-19

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. The distance between the two signal points is $d = \sqrt{1^2 + 2^2} = \sqrt{5}$ and then the error probability is

$$P_\mathrm{e} = Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{5}{2}}\right) \approx Q(1.58) \approx 5.7053 \cdot 10^{-2}.$$

b. The codewords are generated by multiplying G with all length-two binary vectors. This generates the codewords $0\,0\,0\,0\,0$, $1\,1\,1\,0\,0$, $0\,1\,0\,1\,1$, and $1\,0\,1\,1\,1$.

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2

The answer should give a Hamming codes, as described in Section 8.5.3 in the book. Here are some main points that give one point each:

- The dimension of the code is $k = 2^m - m - 1$ and the length of the code is $n = 2^m - 1$.
- The minimum distance is 3, since all columns of the parity check matrix are different but one can find many combinations of three columns that are linearly dependent.
- Hamming codes satisfy the Hamming bound and are therefore called perfect codes.
- The parity-check matrix for $m = 3$, which are given in Example 8.16.
- The generator matrix for $m = 3$, which are given in Example 8.16.

3

a. **False**, the symbol time is inversely proportional to the bandwidth.

b. **True**, that is exactly what it does.

c. **True**, the codes are designed for error detection.

d. **False**, it is the phase that carries the information.

e. **False**, inter-symbol interference appears when the Nyquist condition is not satisfied.

Problem part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

a. The code consists of the two codewords $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$. This is obviously a linear code. The minimum distance is then the smallest non-zero weight among the codewords, which equally obviously is n. The error-correction capability of the code is in general given by

$$t = \left\lfloor \frac{d-1}{2} \right\rfloor .$$

Since $d = n$ is odd, this simplifies to

$$t = \frac{n-1}{2}.$$

b. The size of the code (number of codewords) is 2. The dimension of the code is then

$$k = \log_2(2) = 1.$$

To form a generator matrix, we need one non-zero codeword. The only one is $(1, 1, \dots, 1)$. So, we have the generator matrix

$$G = (1, 1, \dots, 1).$$

G is on systematic form, where we interpret the first one as a 1×1 identity matrix, and the rest of G as the parity part, P. Using the standard trick to find a parity check matrix, we then find

$$H = (P^T, I_{n-1}) = \begin{pmatrix} 1 & 1 & & 0 \\ \vdots & & \ddots & \\ 1 & 0 & & 1 \end{pmatrix}.$$

c. In a, we found that the error correction capability of the code is $t = \frac{n-1}{2}$. Therefore, form decoding spheres of radius t around each codeword. For a perfect code, there are no vectors that are outside such decoding spheres. Thus, we search for a vector that is just outside one of the decoding spheres, that at the same time is outside the other decoding sphere. Consider a vector $\bar{v}$ that is on distance

$$t + 1 = \frac{n+1}{2}$$

from one of the codewords, i.e. just outside the decoding sphere of that codeword. Without loss of generality, let that codeword be $(0, 0, \ldots, 0)$. The vector $\bar{v}$ then contains

$$t + 1 = \frac{n+1}{2}$$

ones and

$$n - (t + 1) = \frac{n-1}{2}$$

zeros. Consequently, $\bar{v}$ is in the decoding sphere of $(1, 1, \ldots, 1)$. In the same way, a vector just outside the decoding sphere of $(1, 1, \ldots, 1)$ is in the decoding sphere of $(0, 0, \ldots, 0)$. We have thus showed that a binary repetition code of odd length is perfect. **Alternative proof:** It is also possible to base the argument on the observation that there are totally

$$\sum_{w=0}^{n} \binom{n}{w} = 2^n$$

vectors in the n-dimensional vector space, and that the size of each decoding sphere is

$$\sum_{w=0}^{(n-1)/2} \binom{n}{w} = 2^{n-1}.$$

The last equality stems from the fact that the sequence of binomial coefficients obey the symmetry rule

$$\binom{n}{w} = \binom{n}{n-w}$$

and that the last sum adds the first half of that sequence, and thus that sum is exactly half the first sum.

Answer:

a. $t = \frac{n-1}{2}$.

b. $H = \begin{pmatrix} 1 & 1 & & 0 \\ \vdots & & \ddots & \\ 1 & 0 & & 1 \end{pmatrix}$

c. —

5

a. For a CRC code, the message $m(x)$ is a polynomial of degree up to $k - 1$. In this case, $k - 1 = 5$ and hence $k = 6$.
The CRC polynomial has always the degree $n-k$. Since the degree in this case is $4 = n-k$, we get $n = 4+k = 10$.

b. The first step in the encoding is to compute $x^{n-k}m(x)$ which in this case is $x^8 + x^6 + x^4$. Next, we should divide $x^{n-k}m(x)$ by the CRC polynomial, which leads to the remainder $r(x) = x^3 + x^2 + x$.
Finally, the codeword is formed as $x^{n-k}m(x) + r(x) = x^8 + x^6 + x^4 + x^3 + x^2 + x$.

Answer: **a.** Length: $n = 10$. Dim.: $k = 6$.
b. The codeword $c(x) = x^8 + x^6 + x^4 + x^3 + x^2 + x$.

6

a. If we assume high SNR, we can use the nearest neighbour approximation.

Scheme I is equivalent to 8-PSK and the smallest distance between signal points are $d_{\min} = 2A_1 \sin\left(\frac{\pi}{8}\right)$. Each constellation point has two neighbours at this distance and the symbol error probability is then $P_e \approx 2Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$

Scheme II has the smallest distance between signal points in the inner circle and it is $d_{\min} = \frac{2A_2}{5}$. Half of the constellation points have two neighbours at this distance and the symbol error probability is then $P_e \approx Q\left(\frac{d_{\min}}{\sqrt{2N_0}}\right)$

b. If we set

$$2Q\left(\frac{2A_1 \sin\left(\frac{\pi}{8}\right)}{\sqrt{2N_0}}\right) = Q\left(\frac{\frac{2A_2}{5}}{\sqrt{2N_0}}\right)$$

we can get the relationship

$$A_2 = Q^{-1}\left(2Q\left(\frac{2A_1 \sin\left(\frac{\pi}{8}\right)}{\sqrt{2N_0}}\right)\right)5\sqrt{\frac{N_0}{2}}.$$

c. If the two constellations should have the same average energy, we need

$$A_1^2 = \frac{1}{2}\left(A_2^2 + A_2^2\frac{2}{25}\right) = A_2^2\frac{27}{50}.$$

If we compare the minimum distances, we notice that

$$\frac{2A_1 \sin\left(\frac{\pi}{8}\right)}{\frac{2A_2}{5}} = 5\sqrt{\frac{27}{50}}\sin\left(\frac{\pi}{8}\right) \approx 1.4$$

which means that Scheme I has a 1.4 times larger minimum distance.

We need to be careful when comparing the symbol error probabilities, because Scheme I has a factor 2 in front of the Q-function, while Scheme II has not. However, at high SNR, the Q-function goes to zero very quickly (much faster than linear) and thus it is better to have a larger $d_{\min}$ than a smaller factor in front of the Q-function. In conclusion, Scheme I will provide the lowest error probability when the SNR is high.

Answer: -

7

This task, at least part a, can be solved in many ways, since the parity check code has more than one generator matrix, and since the task does not specify the order of the bits.

a. First of all, we need generator matrices of the two component codes. Therefore, let G_i be a generator matrix of component code $\mathcal{C}_i$. The given codes are a simple parity check code $\mathcal{C}_1$ with parameters $[4, 3, 2]$ and a repetition code $\mathcal{C}_2$ with parameters $[4, 1, 4]$. Then we have the following possible generator matrices:

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}.$$

Let us consider three simpler cases:

- First we consider something that we could call a $(\bar{u}, \bar{u})$ construction, i.e. we repeat the same codeword $\bar{u} \in \mathcal{C}_1$ twice. Then we create a code with length $2n$ and dimension k_1, and a generator matrix of this code is obviously

  $$(G_1, G_1),$$

  since the same k_1 information bits determine both the first and the last n bits. The minimum distance is $2d_1$, which should be easily realized.

- Then we consider something that we could call a $(\bar{u}+\bar{v})$ construction, where we let a codeword be the sum of $\bar{u} \in \mathcal{C}_1$ and $\bar{v} \in \mathcal{C}_2$. Then we create a code of length n and dimension $k_1 + k_2$, and a generator matrix of this code is obviously

$$\begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

since the n codeword bits are given as the sum of the two involved codewords. There is no general rule that determines the minimum distance in this case.

- Finally we consider something we could call a $(\bar{u}, \bar{v})$ construction, where we let a codeword consist of $\bar{u} \in \mathcal{C}_1$ followed by $\bar{v} \in \mathcal{C}_2$. This is a code of length $2n$ and dimension $k_1 + k_2$, and a generator matrix of this code is just as obviously

$$\begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix},$$

since the first k_1 information bits determine the first n codeword bits, and the following k_2 information bits determine the last n codeword bits. The minimum distance is $\min\{d_1, d_2\}$, which also should be easily realized.

Back to the $(\bar{u}, \bar{u}+\bar{v})$ construction. The first k_1 information bits affect both the first and the last n codeword bits according to the first case. The following k_2 information bits affect only the last n codeword bits, and the combination can be described either in terms of case 2 or in terms of case 3. In any case, we get the generator matrix

$$G = \begin{pmatrix} G_1 & G_1 \\ 0 & G_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix},$$

which as we noted is not the only possible answer.

b. We want to determine the minimum distance of the code. The code is linear, so the minimum distance is also the smallest nonzero weight. We generate all non-zero codewords:

$\overline{m}$	$\bar{c} = \overline{m}G$	$w_{\mathrm{H}}(\bar{c})$
(0001)	(00001111)	4
(0010)	(00110011)	4
(0011)	(00111100)	4
(0100)	(01010101)	4
(0101)	(01011010)	4
(0110)	(01100110)	4
(0111)	(01101001)	4
(1000)	(10011001)	4
(1001)	(10010110)	4
(1010)	(10101010)	4
(1011)	(10100101)	4
(1100)	(11001100)	4
(1101)	(11000011)	4
(1110)	(11111111)	8
(1111)	(11110000)	4

The smallest weight, and thus the smallest distance, is 4.

Note 1: This is the code we get if we start with a Hamming-(7,4) code and append a parity bit such that all codewords have even weight.

Note 2: It can be shown that a code that is created using the $(\bar{u}, \bar{u}+\bar{v})$ construction has minimum distance $d = \min\{2d_1, d_2\}$. In this particular case, we have $2d_1 = d_2 = 4$.

Answer:

a. $G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$

b. $d = 4$.