TSKS01 Digital Communication

Solutions for the exam 2017-04-18

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. The parity-check matrix is

$$H = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

b. The distance between the two signal points is

$$d = \sqrt{(2-0)^2 + (0+2)^2} = \sqrt{2} \cdot 2.$$

Hence, the error probability is given by

$$P_{\mathrm{e}} = Q\left(\frac{d}{\sqrt{2N_0}}\right) \approx Q(2) \approx 2.2750 \cdot 10^{-2}.$$

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2

The answer should give a reasonable description of syndrome decoding Here are some main points that give one point each:

- Dimension of parity check matrix and received signal
- Explanation of what happens when a received signal with error is multiplied with the parity-check matrix.
- Explanation of how to identify the error from the syndrome.
- Motivation of why syndrome decoding leads to an ML detector
- An example of syndrome decoding with numbers in the matrices/vectors.

3

a. True, since a constant frequency response leads to an impulse response that is a (scaled) delta-function, thus you have an AWGN channel.

b. True, because if we guess that every bit is one then we get a bit error probability of 1/2. ML detection cannot be worse than that.

c. True, these codes are used to get few bit errors for each symbol error.

d. False, only repetition codes of odd length satisfy the Hamming bound.

e. False, the product is zero.

Problem part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

First, we determine the distance d between the signals:

$$\begin{aligned} d^2 &= \int_0^{T_\mathrm{b}} \left(s_1(t) - s_2(t)\right)^2 \, dt \\ &= \int_{aT_\mathrm{b}}^{T_\mathrm{b}} A^2 \, dt = (1-a)A^2 T_\mathrm{b}. \end{aligned}$$

Let E_1 and E_2 be the energies of the two signals. Then the average signal energy is given by

$$\begin{aligned} E_\mathrm{b} &= \frac{1}{2}(E_1 + E_2) = \frac{1}{2}\left(\int_0^{T_\mathrm{b}} A^2 \, dt + \int_0^{aT_\mathrm{b}} A^2 \, dt\right) \\ &= \frac{1+a}{2} A^2 T_\mathrm{b}, \end{aligned}$$

since the two signals are equally probable. This gives us

$$A^2 T_\mathrm{b} = \frac{2E_\mathrm{b}}{1+a},$$

which used in the expression of d^2 above gives us

$$d^2 = \frac{1-a}{1+a} 2E_\mathrm{b}.$$

The error probability is as usual for binary modulation schemes given by

$$P_\mathrm{b} = Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{1-a}{1+a} \cdot \frac{E_\mathrm{b}}{N_0}}\right) \leq 10^{-5}.$$

The Q table gives us

$$\sqrt{\frac{1-a}{1+a} \cdot \frac{E_\mathrm{b}}{N_0}} \geq 4.26,$$

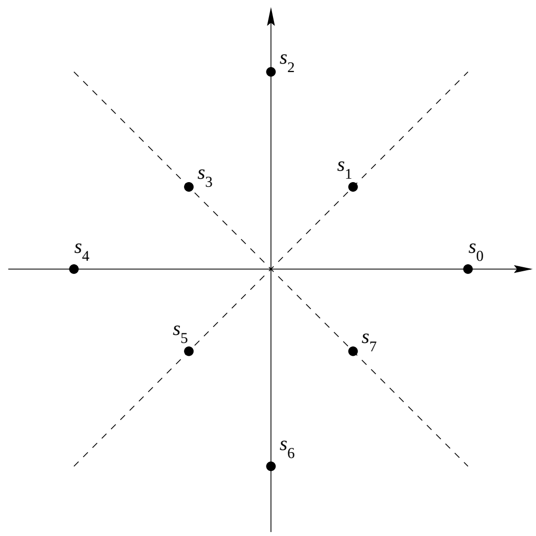
which, using the given range $0 \le a \le 1$, we can rewrite as

$$a \le \frac{\frac{E_b}{N_0} - 4.26^2}{\frac{E_b}{N_0} + 4.26^2} = 0.3738.$$

Answer: $0 \le a \le 0.3738$.

5

We were given the following constellation:



We have high SNR. Therefore, it is enough to maximize the minimum distance of the constellation. Assume that we have $E_0 \ge E_1$. Based on the symmetry of the constellation, it is enough to consider the quadratic distances

$$d_1^2 = |s_2 - s_1|^2 = E_2 + E_1 - 2\sqrt{E_2 E_1/2} \quad \text{and}$$
$$d_2^2 = |s_3 - s_1|^2 = E_3/2 + E_1/2 + 2\sqrt{E_3 E_1/4}.$$

By using the given demands $E_0 = E_2 = E_4 = E_6$ and $E_1 = E_3 = E_5 = E_7$ we get

$$d_1^2 = E_0 + E_1 - 2\sqrt{E_0 E_1/2} \quad \text{and}$$
$$d_2^2 = 2E_1.$$

Furthermore, we demand that the average energy, say E, is constant. Thus, we have

$$E = \frac{1}{2}(E_0 + E_1) \qquad \Rightarrow \qquad E_0 = 2E - E_1$$

This gives us

$$d_1^2 = 2E - 2\sqrt{E_1(2E - E_1)/2} \quad \text{and}$$
$$d_2^2 = 2E_1.$$

There are three interesting cases, and those are $d_1 < d_2$, $d_1 = d_2$ and $d_1 > d_2$. Let us start by considering the case $d_1 = d_2$. Then the two equations above give us

$$E - \sqrt{E_1(2E - E_1)/2} = E_1 \quad \Rightarrow \tag{1}$$
$$E_1^2 - 2EE_1 + \frac{2}{3}E^2 = 0 \quad \Rightarrow$$
$$E_1 = \left(1 \pm \frac{1}{\sqrt{3}}\right) E.$$

It is only the solution $E_1 = \left(1 - \frac{1}{\sqrt{3}}\right) E$ that fulfills Equation ??. Obviously, for smaller E_1 then we have $d_1 > d_2$ and d_2 is the minimum distance. Equally obviously, for larger $E_1 \leq E$ then we have $d_1 < d_2$ and d_1 is the minimum distance. Let us study the quadratic minimum distance $d^2 = \min\{d_1^2, d_2^2\}$. Based on the above, in the interval $0 \leq E_1 \leq \left(1 - \frac{1}{\sqrt{3}}\right) E$ we have

$$d^2 = 2E_1$$

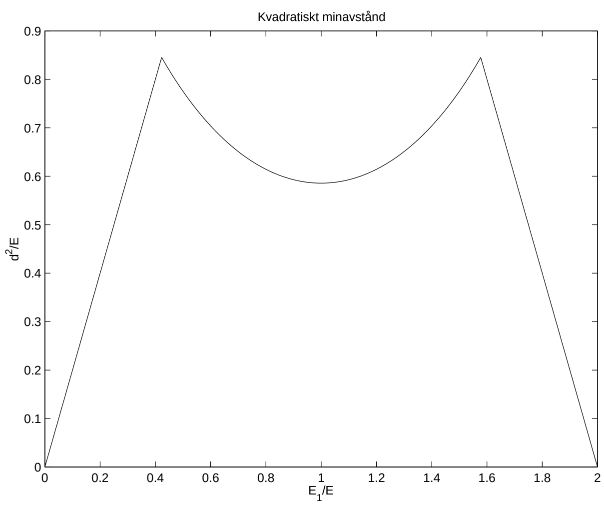
and in the interval $\left(1 - \frac{1}{\sqrt{3}}\right) E < E_1 < \left(1 + \frac{1}{\sqrt{3}}\right) E$ we have

$$d^2 = 2E - 2\sqrt{E_1(2E - E_1)/2}.$$

and in the interval $\left(1 - \frac{1}{\sqrt{3}}\right) E \leq E_1 \leq 2E$ we have

$$d^2 = 4E - 2E_1,$$

where we have used symmetry for the interval $E \leq E_1 \leq 2E$. Formally we should differentiate and reason. Instead we draw a graph.



The first maximum is found for $d_1 = d_2$, i.e. for

$$E_1 = \left(1 - \frac{1}{\sqrt{3}}\right) E$$

and the corresponding minimum distance is

$$d_{\text{opt}}^2 = 2\left(1 - \frac{1}{\sqrt{3}}\right) E.$$

The second maximum in the graph corresonds to the case where E_0 and E_1 have switched places. Then we have

$$E_1 = \left(1 + \frac{1}{\sqrt{3}}\right) E$$

For 8-PSK we have the quadratic minimum distance

$$d_{\text{PSK}}^2 = 4E \sin^2(\pi/8).$$

Thus, we have the quotient $d^2_{\text{opt}}/d^2_{\text{PSK}} = 1.443$, which corresponds to 1.59 dB.

Answer: $E_1 = \left(1 \pm \frac{1}{\sqrt{3}}\right) E$ and $E_2 = \left(1 \mp \frac{1}{\sqrt{3}}\right) E$.
This is 1.59 dB better than 8-PSK with the same average energy.

6

a. This linear shift register is given in Figure 8.7 in the book (edition 2016).

b. The message is $m(x) = x^4 + x^3 + x$ and the CRC polynomial is $p(x) = x^4 + x^3 + x + 1$. Note that $n - k = 4$ and that

$$x^4 m(x) = (x^4 + 1)p(x) + x^3 + x + 1$$

which can be seen directly or by performing polynomial division. Hence $r(x) = x^3 + x + 1$ and the codeword is

$$x^4 m(x) + r(x) = x^8 + x^7 + x^5 + x^3 + x + 1.$$

Answer: -

7

a. The length n of the code is the number of columns in G, thus $n = 7$. The dimension k of the code is the number of rows in G, thus $k = 4$. The size M of the code is as usual given by $M = 2^k = 16$. There are many different ways to determine the minimum distance d. If we compute the parity-check matrix

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

we notice that the minimum number of linearly dependent columns is 3. Hence, $d = 3$.

b. With $r = (0111010)$, the syndrome is $s^T = Hr^T = (110)^T$. This is equal to the third column in the parity-check matrix, thus ML decoding gives $C = (0101010)$.

c. We have $2^n = 2^7$ and $2^k \sum_{i=0}^{1} \binom{n}{i} = 2^4(1+7) = 2^7$. Since these terms are equal, the Hamming bound is satisfied.

Answer: a. Length: $n = 7$. Dim.: $k = 4$. Size: $M = 16$. Min. dist.: $d = 3$.
b. The codeword is (0101010).
c. Hamming bound is satisfied.