TSKS01 Digital Communication

Solutions for the exam 2017-01-09

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Introductory task

As partial fulfillment to pass the exam, you have to solve at least one of these two subtasks correctly.

1

a. The codewords are (00000), (01011), (10110), (11101).

b. We have the two signal points

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} \text{and} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The distance between those points is

$$d = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

Finally, the error probability is given by

$$P_\mathrm{e} = Q\left(\frac{d}{\sqrt{2N_0}}\right) \approx Q(1) \approx 1.5866 \cdot 10^{-1}.$$

Question part

The questions in this part can give you at most 5 points each. You need at least 3 points from this part of the exam to pass.

2

The answer should give a reasonable description of CRC codes and their main properties. Here are some main points that give one point each:

- Description of the CRC polynomial and the representation of message/codeword as polynomials.
- Formulas for generation of a CRC codeword, as described on page 168.
- Description of the detection process, as described on page 168.
- An example of the detection of errors using polynomial division.
- A linear feedback shift register implementation example of the type in Figure 8.6 or Figure 8.7.

3

a. **True**, the sinc pulse has the smallest bandwidth among all pulse functions that satisfy the Nyquist criterion. The excess bandwidth of the raised cosine pulse is shown in Figure 4.18.

b. **False**, the main purpose of the phase-locked loop is to produce a sinusoid with the correct carrier frequency and phase. This sinusoid can be used for demodulation, but has no connection to decoding of error control codes.

c. **False**, the rate of a code is k/n where $k, n \geq 1$ and $k \leq n$. Hence, the rate is between 0 and +1.

d. **True**, the error events that are counted multiple times in the union bound have negligibly low probability at high SNR.

e. **True**, since both diagram contains four points that are the corners of a square.

Problem part

The problems in this part can give you at most 5 points each. You need at least 6 points from this part of the exam to pass.

4

a. The energy of the pulse can be computed as

$$E = 4\int_0^1 (Ax)^2 dx = \frac{4A^2}{3}.$$

The symbol error probability is

$$P_{\mathrm{e}} = Q\left(\sqrt{\frac{2E}{N_0}}\right) = Q\left(\sqrt{\frac{8A^2}{3N_0}}\right).$$

b. From the table of Q function value, we notice that $Q(3.72) \approx 9.9611 \cdot 10^{-5} \approx 10^{-4}$. To make $P_{\mathrm{e}} \approx 10^{-4}$ we thus need $3.72 \approx \frac{8A^2}{3N_0}$, which implies $A \approx 3.72\frac{3}{8} \approx 2.278$.

c. A constant function with $+A/\sqrt{3}$ for $0 \leq t < 4$ (and zero elsewhere) has the same energy, but the smaller peak values.

Answer:

a. $P_{\mathrm{e}} = Q\left(\sqrt{\frac{8A^2}{3N_0}}\right)$.

b. $A \approx 2.278$.

c. One possibility: $+A/\sqrt{3}$ for $0 \leq t < 4$

5

We have only considered binary codes in the course. Therefore this solution will assume that. The following generalizes quite easily to larger alphabets. Recall the definition of a binary linear code: *A binary code is linear if all sums of codewords are codewords.*

a. Consider a codeword $\bar{c}$ in a linear code. According to the definition, $\bar{c} + \bar{c}$ is then also a codeword. But the addition is to be performed componentwise, reduced modulo 2. Thus, we have $\bar{c} + \bar{c} = \bar{0}$. So, the all-zero vector is in the code.

b. For the Hamming distance $d_{\rm H}(\bar{a},\bar{b})$ and the Hamming weight $w_{\rm H}(\bar{c})$, we have for binary vectors,

$$d_{\rm H}(\bar{a},\bar{b}) = w_{\rm H}(\bar{a}+\bar{b}).$$

This is a property of the two measures, and has nothing to do with if we are considering any code at all. Now, consider a code $\mathcal{C}$. Then we have the minimum distance

$$d = \min_{\substack{\bar{c}_1,\bar{c}_2\in\mathcal{C}\\ \bar{c}_1\neq\bar{c}_2}} d_{\rm H}(\bar{c}_1,\bar{c}_2) = \min_{\substack{\bar{c}_1,\bar{c}_2\in\mathcal{C}\\ \bar{c}_1\neq\bar{c}_2}} w_{\rm H}(\bar{c}_1+\bar{c}_2),$$

using the observation above. Now, assuming that $\mathcal{C}$ is linear, then $\bar{c}_1+\bar{c}_2$ is a non-zero codeword in $\mathcal{C}$. Also, if we fix $\bar{c}_1$, and let $\bar{c}_2$ run through all other codewords in $\mathcal{C}$, then $\bar{c}_1+\bar{c}_2$ runs through all non-zero codewords in $\mathcal{C}$. Therefore, we can rewrite the expression above as

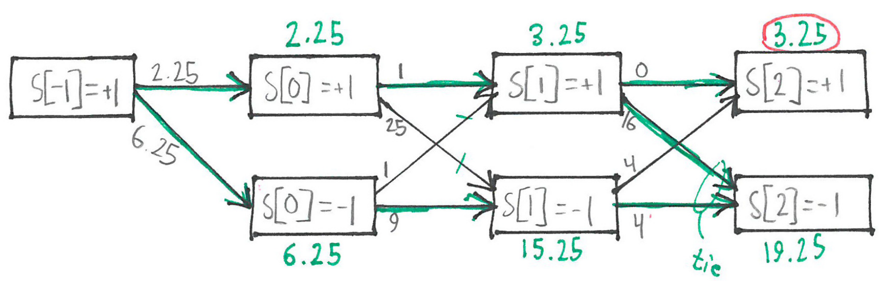
$$d = \min_{\substack{\bar{c}\in\mathcal{C}\\ \bar{c}\neq\bar{0}}} w_{\rm H}(\bar{c}),$$

which is what we were expected to show.

Answer: —

6

The ML estimate of the sequence $(s[0], s[1], s[2])$ can be obtained using the Viterbi algorithm. In this case, the transition metric is $\Lambda_k(s_n[k], \mathcal{S}_n[k]) = (z[k] - 2s[k] + s[k-1])^2$. By computing all the transition metrics and creating a trellis, we obtain:



The smallest accumulated metric is 3.25 and it is achieved by $(s[0], s[1], s[2]) = (1, 1, 1)$.

Alternative problem, only for previous-year students:

According to the MAP detection rule, we should choose the signal point that maximizes the expression

$$\Pr\{S=s_k\}\cdot f_{X|S}(x|s_k),$$

where we have the two signal points $s_0 = \sqrt{E}$ and $s_1 = -\sqrt{E}$. Since N is Gaussian with mean 0 and variance $\sigma_N^2 = N_0/2$, we have

$$f_N(n) = \frac{1}{\sqrt{2\pi}\sigma_N} e^{n^2/2\sigma_N^2} = \frac{1}{\sqrt{\pi N_0}} e^{-n^2/N_0},$$
$$f_{X|S}(x|s_0) = f_N(x-s_0) = \frac{1}{\sqrt{\pi N_0}} e^{-(x-\sqrt{E})^2/N_0},$$
$$f_{X|S}(x|s_1) = f_N(x-s_1) = \frac{1}{\sqrt{\pi N_0}} e^{-(x+\sqrt{E})^2/N_0}.$$

According to the problem formulation we have the probabilities

$$\Pr\{S=s_0\} = p, \qquad \text{and} \qquad \Pr\{S=s_1\} = 1-p.$$

All in all, we are interested in comparing the two expressions

$$p \cdot \frac{1}{\sqrt{\pi N_0}} e^{-(x-\sqrt{E})^2/N_0},$$
$$(1-p) \cdot \frac{1}{\sqrt{\pi N_0}} e^{-(x+\sqrt{E})^2/N_0}.$$

Let z be the value of x for which those two expressions are equal. Then z is the decision border between the two detection regions. I.e. we want to solve the equation

$$p \cdot \frac{1}{\sqrt{\pi N_0}} e^{-(z-\sqrt{E})^2/N_0} = (1-p) \cdot \frac{1}{\sqrt{\pi N_0}} e^{-(z+\sqrt{E})^2/N_0}.$$

Multiplying both sides by $\sqrt{\pi N_0}$ gives us

$$p \cdot e^{-(z-\sqrt{E})^2/N_0} = (1-p) \cdot e^{-(z+\sqrt{E})^2/N_0}.$$

The natural logarithm of those expressions are then

$$\ln(p) - \frac{(z-\sqrt{E})^2}{N_0} = \ln(1-p) - \frac{(z+\sqrt{E})^2}{N_0}.$$

Multiplying both sides by N_0 and expanding the squares give us

$$N_0 \ln(p) - z^2 + 2z\sqrt{E} - E =$$
$$= N_0 \ln(1-p) - z^2 - 2z\sqrt{E} - E$$

Adding $z^2 + E$ to both sides gives us

$$N_0 \ln(p) + 2z\sqrt{E} = N_0 \ln(1-p) - 2z\sqrt{E}.$$

Solving the above for z gives us

$$z = \frac{N_0}{4\sqrt{E}} \ln\left(\frac{1-p}{p}\right).$$

Then we have

$$p \cdot \frac{1}{\sqrt{\pi N_0}} e^{-(x-\sqrt{E})^2/N_0} > (1-p) \cdot \frac{1}{\sqrt{\pi N_0}} e^{-(x+\sqrt{E})^2/N_0}$$

for the case $x > z$ and the estimated symbol should be s_0. Conversely, we have

$$p \cdot \frac{1}{\sqrt{\pi N_0}} e^{-(x-\sqrt{E})^2/N_0} < (1-p) \cdot \frac{1}{\sqrt{\pi N_0}} e^{-(x+\sqrt{E})^2/N_0}$$

for the case $x < z$ and the estimated symbol should then be s_1.

Answer: The ML sequence estimate is $s[0] = 1$, $s[1] = 1$, $s[2] = 1$. The smallest metric is 3.25.

7

The throughput is defined as

$$\frac{\log_2(M)}{T}\frac{k}{n}(1 - P_{\text{packet}}.$$

a. At high SNR, the throughput with uncoded transmission converges to $\frac{\log_2(M)}{T}$, which is $4 \cdot 10^7$ for Technique 5. This implies $\log_2(M) = 4$ and $M = 16$. Hence, 16-QAM could have been used.

b. The generator matrix has $k = 5$ and $n = 10$, thus the coding rate is $k/n = 0.5$. The throughput at high SNR should be $0.5 \cdot 40 = 20$ Mbit/s, which is the case of Technique 3.

c. The coding rate for Technique 1 is 1/8. For a repetition code with $k = 1$, we need $n = 8$. A generator matrix is

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Answer:
a. 16-QAM
b. Technique 3.
c. $n = 8$ and $G = (11111111)$.