Digital Filters 190529

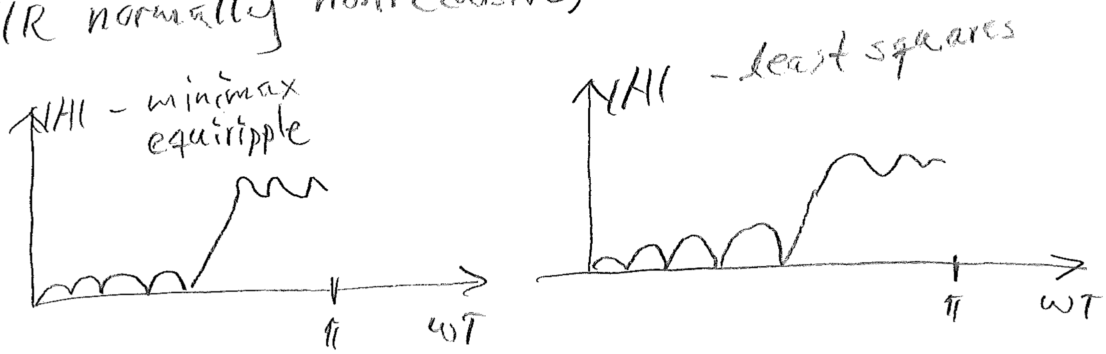
1 a) Group delay: $\tau_g(\omega T) = -\frac{\partial \phi(\omega T)}{\partial \omega}$ $\left(\text{or } -\frac{\partial \phi(\omega T)}{\partial (\omega T)}\right)$

Phase delay: $\tau_p(\omega T) = -\frac{\phi(\omega T)}{\omega}$ $\left(\text{or } -\frac{\phi(\omega T)}{\omega T}\right)$

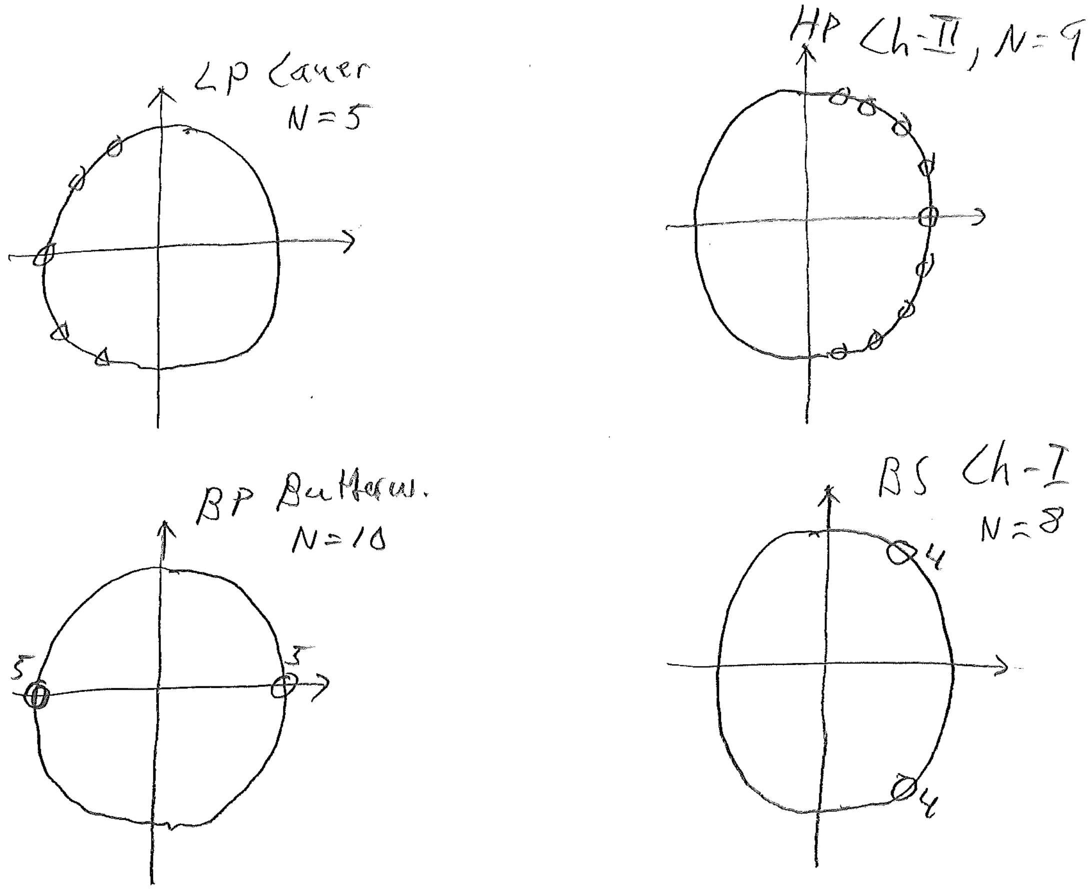
b) Stable filter under finite-arithmetic conditions
Recovers after disturbance, no parasitic oscillations

c) Recursive - output computed using input samples and previous output samples
Nonrecursive - output computed using only input samples
IIR always recursive
FIR normally nonrecursive, can be recursive

d)



e)



2) Edges of analog filter ($\frac{2}{T}=1$ for simplicity)

$\omega_{ac} = \tan\left(\frac{\omega_c T}{2}\right) = 0{,}207 \qquad \omega_{as} = \tan\left(\frac{\omega_s T}{2}\right) = 1$

$\frac{\omega_{as}}{\omega_{ac}} = 4{,}82 \Rightarrow$

$12 \le \theta \le 16$ select $\theta = 14$

Filter order $N = 3$

Normalized zeros: $\pm j4.7852$, one at $s=\infty$

poles: -1.0612

$-0.4633 \pm j1.2119$

Denormalized zeros $\pm j0{,}9848$, one at $s=\infty$

Mult. by ω_{ac} poles $-0{,}2073$

$-0{,}09594 \pm j0{,}2510$

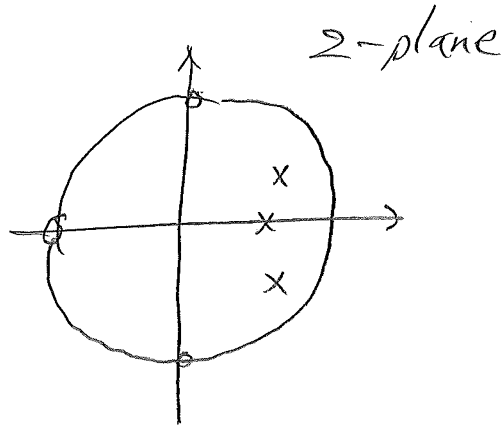
$s \to z = \frac{1+s}{1-s}$:

zeros: $0{,}01536 \pm j0{,}9999$

-1.0000

poles : 0.6565

$0{,}7340 \pm j0{,}3971$



3) Edges of analog reference filter

$\omega_{arc} = \tan\left(\frac{0.8\pi}{2}\right) = 3.078$, $\omega_{ars} = \tan\left(\frac{0.55\pi}{2}\right) = 1.171$

HP → LP spec. $\Omega_{arc} = \frac{\omega_I^2}{\omega_{arc}} = \left| \text{select } \omega_I^2 = \omega_{arc} \right| = 1$

$$\Omega_{ars} = \frac{\omega_I^2}{\omega_{ars}} = \frac{\omega_{arc}}{\omega_{ars}} \approx 2.629$$

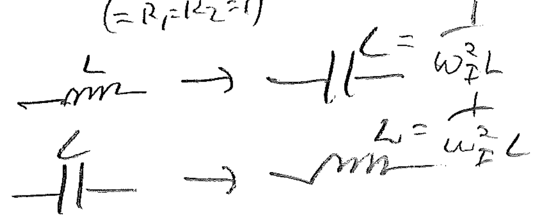
$\frac{\Omega_{ars}}{\Omega_{arc}} = 2{,}629 \Rightarrow$ Filter order $N = 5$

Normalized elem. values $L_1' = L_5' = 1.7058$, $L_3' = 2.5408$

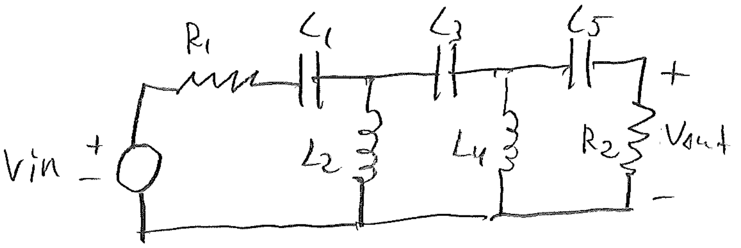
$C_2' = C_4' = 1.2296$

Denormalize with $R = 1$, $\Omega_{arc} = 1 \Rightarrow L = L'$, $C = C'$
$(= R_1 = R_2 = 1)$

LP → HP :



$\Rightarrow$

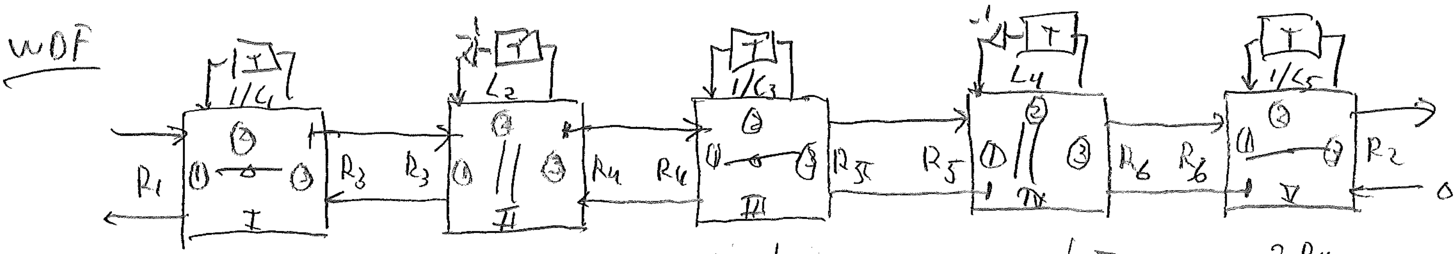


$L_2 = L_4 = \frac{1}{\omega_I^2 C_2} = 0{,}2642$

$C_1 = C_5 = \frac{1}{\omega_I^2 L_1} = 0{,}1905$

$C_3 = \frac{1}{\omega_I^2 L_3} = 0{,}1279$, $R_1 = R_2 = 1$

WDF



I: $R_3 = R_1 + \frac{1}{C_1} = 6{,}2499$

$\alpha_1 = \frac{2R_1}{R_1 + 1/C_1 + R_3} = 0{,}1600$

$\alpha_2 = 1 - \alpha_1 = 0.8400$

$\alpha_3 = 1$

II : $R_4 = \frac{1}{1/R_3 + 1/L_2} = 0{,}2535$

$\alpha_1 = \frac{2/R_3}{1/R_3 + 1/R_2 + 1/R_4} = 0{,}04057$

$\alpha_2 = 1 - \alpha_1 = 0{,}95943$

$\alpha_3 = 1$

III: $\alpha_1 = \frac{2 R_4}{R_4 + 1/C_3 + R_5} = 0{,}06089$

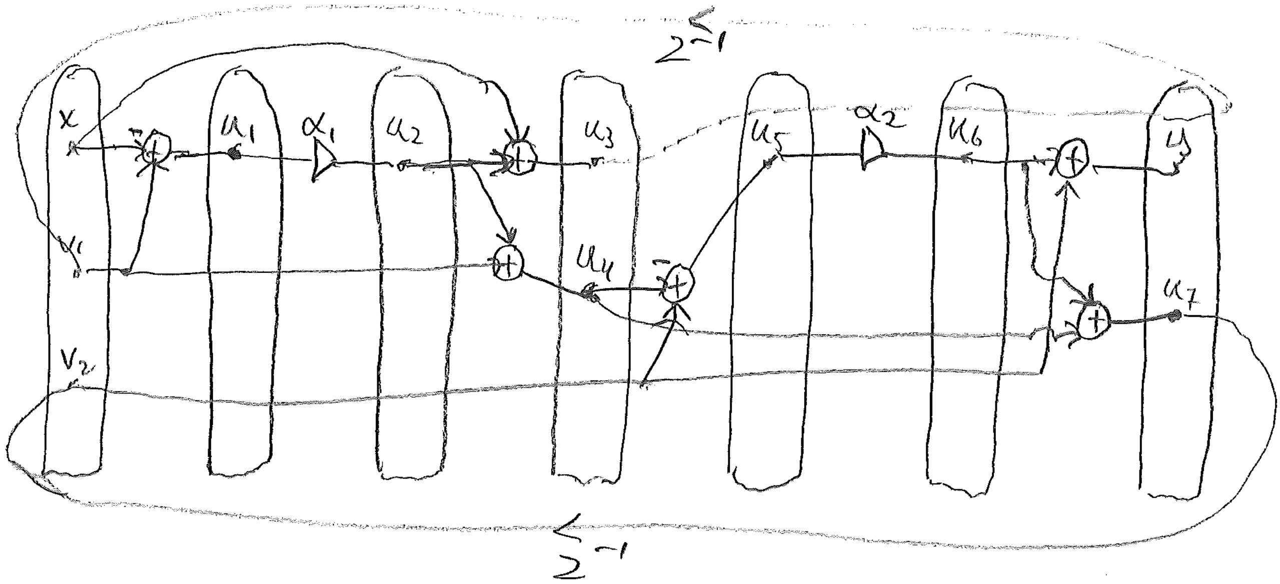
$\alpha_2 = 2 - \alpha_1 - \alpha_3 = 1.8782$

$\alpha_3 = \alpha_1 = 0{,}06089$
$R_4 = R_5$

IV : $R_5 = R_4$, $\alpha_1 = 1$
(Symm. w. II)
$\alpha_2 = 0.95943$
$\alpha_3 = 0{,}04057$

V: $R_6 = R_3$, $\alpha_1 = 1$
(Symmetric to I with I)
$\alpha_2 = 0{,}7520$
$\alpha_3 = 0{,}2480$

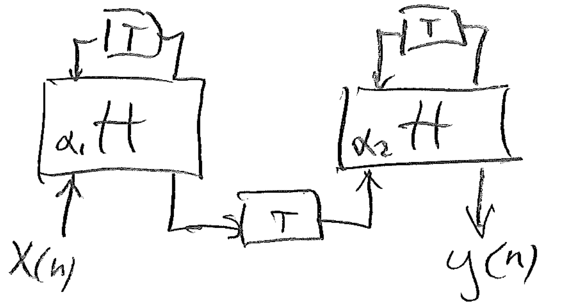
4a)



b) $T_{CP} = 4T_{add} + 2T_{mult}$

$T_{min} = 2T_{add} + T_{mult}$

c)



$\Rightarrow T_{CP} = 2T_{add} + T_{mult} = T_{min}$

5 a) Propagation through allpass does not affect the noise power $\Rightarrow$

$$P_{ye} = \frac{Q^2}{12}\cdot\left(\sum_{n=0}^{\infty} g_1^2(n) + g_2^2(n)\right), \qquad \left.\begin{aligned} g_1(n) &= \alpha_1^n u(n) + \alpha_1^{n-1}u(n-1)\\ g_2(n) &= \alpha_2^n u(n) + \alpha_2^{n-1}u(n-1)\end{aligned}\right\} (1)$$

$$\sum_{n=0}^{\infty} g_1^2(n) = 1 + \left(1+\frac{1}{\alpha_1}\right)^2 \sum_{n=1}^{\infty} \alpha_1^{2n} = 1 + \frac{\alpha_1^2+2\alpha_1+1}{\cancel{\alpha_1^2}}\cdot\frac{\cancel{\alpha_1^2}}{1-\alpha_1^2} = \frac{2(1+\alpha_1)}{1-\alpha_1^2} = \frac{2}{1-\alpha_1}$$

$$\approx 2,667$$

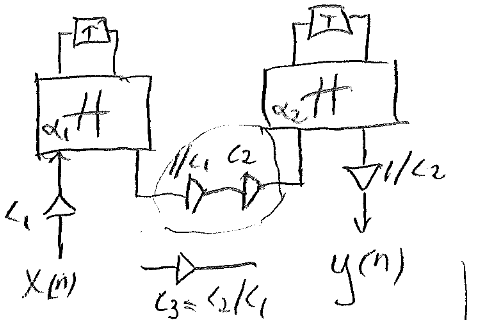
$$\sum_{n=0}^{\infty} g_2^2(n) = \dots = \frac{2}{1-\alpha_2^2} \approx 8 \qquad \Rightarrow P_{ye} \approx 5.298\cdot 10^{-8}$$

$$(1): Y(z) = E(z) + (1+\alpha)V(z) = \Big/ V(z) = z^{-1}\left(E(z)+\alpha V(z)\right) \Rightarrow V(z) = E(z)\cdot\frac{z^{-1}}{1-\alpha z^{-1}} \Big/$$

$$= E(z)\cdot\left(1+\frac{(1+\alpha)z^{-1}}{1-\alpha z^{-1}}\right) = E(z)\cdot\frac{1+z^{-1}}{1-\alpha z^{-1}} \longleftrightarrow \alpha^n u(n) + \alpha^{n-1}u(n-1)$$

---

b) L_2-norm not affected by allpass sections $\Rightarrow$



$$\left.\begin{aligned} f_1(n) &= -\alpha_1^n u(n) + \alpha_1^{n-1}u(n-1)\\ f_2(n) &= -\alpha_2^n u(n) + \alpha_2^{n-1}u(n-1)\end{aligned}\right\} (2)$$

$$\sum_{n=0}^{\infty} f_1^2(n) = 1 + \left(-1+\frac{1}{\alpha_1}\right)^2 \sum_{n=1}^{\infty}\alpha_1^{2n} = 1 + \frac{\alpha_1^2-2\alpha_1+1}{\cancel{\alpha_1^2}}\cdot\frac{\cancel{\alpha_1^2}}{1-\alpha_1^2}$$

$$= \frac{2(1-\alpha_1)}{1-\alpha_1^2} = \frac{2}{1+\alpha_1} = 1.6$$

$$\sum_{n=0}^{\infty} f_2^2(n) = \dots = \frac{2}{1+\alpha_2} = 1,1428$$

$$c_1 = \frac{1}{\sqrt{1.6}} \approx 0.7906$$

$$c_2 = \frac{1}{\sqrt{1.1428}} \approx 0.9354, \quad c_3 = 1,1832, \quad 1/c_2 = 1.069$$

$$(2): \underset{\text{Input to mult}}{U(z)} = \overset{\text{Del. el. output}}{V(z)} - X(z) = z^{-1}\left(\alpha U(z) + X(z)\right) - X(z)$$

$$\Rightarrow U(z) = X(z)\cdot\frac{-1+z^{-1}}{1-\alpha z^{-1}} \longleftrightarrow -\alpha^n u(n) + \alpha^{n-1}u(n-1)$$

6a) $H(z) = H_1(z^{L_2})\, H_2(z)$

b) Transition band of $H_1(z)$: $\Delta_1 = 2\Delta L_2$

$H_2(z)$: $\Delta_2 = 2\pi\left(\frac{1}{L_2} - \frac{1}{L}\right)$

complexity – mult. per output sample

$$C = \frac{K}{\Delta_1 \cdot L} + \frac{K}{\Delta_2 \cdot L_2} = \frac{K}{2\Delta L_2 \cdot L} + \frac{K}{2\pi\left(1 - \frac{L_2}{L}\right)}$$

$$\frac{\partial C}{\partial L_2} = 0 \quad \text{for } L_2 = -\frac{a}{2} \underset{(-)}{+} \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

where $a = \dfrac{-4\pi}{\frac{2\pi}{L} - 2\Delta L}$, $b = \dfrac{2\pi L}{\frac{2\pi}{L} - 2\Delta L}$

$L = 18 \Rightarrow L_2^{(opt)} \approx 10.25$

c) select $L_1 = 2$, $L_2 = 9 \Leftrightarrow L = L_1 L_2 = 18$

7) $$H(z) = \frac{(1 - z^{-1})(1 + bz^{-1})(1 + b^2 z^{-2})}{(1 - bz^{-1})(1 + bz^{-1})(1 + b^2 z^{-2})}$$

$$= \frac{(1 + (b-1)z^{-1} - bz^{-2})(1 + b^2 z^{-2})}{1 - b^4 z^{-4}}$$

$$= \frac{1 + (b-1)z^{-1} + (b^2 - b)z^{-2} + b^2(b-1)z^{-3} - b^3 z^{-4}}{1 - b^4 z^{-4}} = \sum_{k=0}^{3} z^{-k} H_k(z^4)$$

$$\Longrightarrow \quad H_0(z) = \frac{1 - b^3 z^{-1}}{1 - b^4 z^{-1}}, \quad H_1(z) = \frac{b-1}{1 - b^4 z^{-1}}$$

$$H_2(z) = \frac{b^2 - b}{1 - b^4 z^{-1}}, \quad H_3(z) = \frac{b^2(b-1)}{1 - b^4 z^{-1}}$$