TSKS04 Digital Communication

Continuation Course

Lecture 5

Phase and Delay Estimation, Hypothesis testing

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Recall: Proper Complex Gaussian

Definition: The $N \times 1$ complex random vector $\boldsymbol{x} = \boldsymbol{x}_c + j\boldsymbol{x}_s$ is *proper complex Gaussian* if the real vectors $\boldsymbol{x}_c, \boldsymbol{x}_s$ are jointly Gaussian and

$$E\{(\boldsymbol{x} - E\{\boldsymbol{x}\})(\boldsymbol{x} - E\{\boldsymbol{x}\})^T\} = 0$$

Notation:

- Mean value: $\boldsymbol{m}_x = E\{\boldsymbol{x}\}$
- Covariance matrix: $\boldsymbol{C}_x = E\{(\boldsymbol{x} - E\{\boldsymbol{x}\})(\boldsymbol{x} - E\{\boldsymbol{x}\})^H\}$
- PDF:

$$p(\boldsymbol{x}) = \frac{1}{\pi^N \det(\boldsymbol{C}_x)} e^{-(\boldsymbol{x}-\boldsymbol{m}_x)^H \boldsymbol{C}_x^{-1} (\boldsymbol{x}-\boldsymbol{m}_x)}$$

Notation: $\boldsymbol{x} \sim CN(\boldsymbol{m}_x, \boldsymbol{C}_x)$

Complex White Gaussian Noise (WGN)

Definition: A stochastic process $n(t)$ is complex white Gaussian noise (WGN) if $n(t) = n_c(t) + jn_s(t)$, where $n_c(t), n_s(t)$ are independent and identically distributed (i.i.d.) real WGN.

Property: A Complex WGN is a zero mean, proper complex Gaussian process with ACF

$$C_n(t_1, t_2) = 2\sigma^2 \delta(t_1 - t_2)$$

PSD of real and imaginary part ($N_0 = 2\sigma^2$)

Property: Let $s(t)$ be a complex-valued signal and $n(t)$ is complex WGN, then

$$Z = < n, s > \sim CN(0,2||s||^2\sigma^2)$$

Proofs: Section 4.5.1

Receiver Synchronization (1/2)

Transmitted complex baseband signal:

- $u(t) = \sum_m b[m] g_{TX}(t - mT)$

Passband signal at carrier frequency f_c:

- $u_p(t) = Re\{u(t)e^{j2\pi f_c t}\}$

Non-dispersive AWGN channel, impulse response: $A\delta(t-\tau)$

- $y_p(t) = Au_p(t-\tau) + n_p(t) = Re\{Au(t-\tau)e^{j2\pi f_c t}\underbrace{e^{-j2\pi f_c \tau}}_{e^{j\theta}}\} + n_p(t)$

Received complex baseband signal (downconverted with $f_c - \Delta f$):

- $y(t) = Ae^{j\underbrace{(2\pi\Delta f+\theta)}_{}}u(t-\tau) + n(t)$

Complex AWGN

Channel gain

Carrier offset

Delay

Receiver Synchronization (2/2)

- Acquire channel gain A ← Called "gain" but is really an attenuation
  - Channel estimation:
- Timing synchronization
  - Estimate delay τ: ← At order of symbol times
- Carrier synchronization
  - Estimate carrier offset Δf
  - Estimate phase $\theta = -j2\pi f_c \tau$ ← Large f_c: Essentially uniform distribution $[0,2\pi)$

Principle: Estimate using training symbols

Signals in Known Subspace (1/2)

Recall: K observations of a signal in noise:

$$\boldsymbol{y} = \theta\mathbf{1} + \boldsymbol{w}$$

- Signal "lives" in the subspace $\mathbf{1}$
- Noise is spread in all K dimensions
- MVU estimator: $\hat{\theta}(\boldsymbol{y}) = \mathbf{1}^T\boldsymbol{y}/\|\mathbf{1}\| = \sum_{k=1}^{K} \frac{y[k]}{K}$

Principle: Project $\boldsymbol{y}$ onto signal subspace!

This is a general principle that we have seen before:
Signals along different orthogonal basis functions can be treated separately!

Signals in Known Subspace (2/2)

Consider a noisy signal modeled in complex baseband as

$$y(t) = s(t)f(\theta) + n(t)$$

- θ: Unknown parameter
- $f(\cdot)$: A known function
- $s(t)$: Known complex-valued signal
- $n(t)$: Complex white Gaussian noise with PDF $N_0/2$ per real dim

To estimate θ it is sufficient to consider $z = < y, s >$ ← Called: sufficient statistic

The log-likelihood function is

$$p(z|\theta) = \frac{1}{\pi \|s\|^2 N_0} e^{-\frac{1}{\|s\|^2 N_0}\left|z - \|s\|^2 f(\theta)\right|^2}$$

Parameter Estimation with Hypotheses

Hypothesis Testing

- Goal: Guess properties of a signal
- M hypothesis $H_0, \dots, H_{M-1}$
- Bayesian modeling: $\Pr\{H_i\}$ for $i = 0, \dots, M-1$
- Select the "best" hypothesis based on an observation

Applications in Digital Communication

- Detect which constellation point was sent
- Further application: Estimate delay, phase, and carrier offset

Example: Two Signals in Noise

Additive noise channel:

$$Y = S + W$$

- S is 0 or 1, equal probability
- $W \sim N(0, \sigma_W^2)$ is independent Gaussian noise

H$_0$: $S = 0$, thus $Y|S \sim N(0, \sigma_W^2)$ and likelihood

$$p(y|0) = \frac{1}{\sqrt{2\pi}\sigma_W} e^{-\frac{y^2}{2\sigma_W^2}}$$

H$_1$: $S = 1$, thus $Y|S \sim N(1, \sigma_W^2)$ and likelihood

$$p(y|1) = \frac{1}{\sqrt{2\pi}\sigma_W} e^{-\frac{(y-1)^2}{2\sigma_W^2}}$$

Likelihood ratio test:

$$L(y) = \frac{p(y|1)}{p(y|0)}$$

$L(y) > 1$: H_1 most likely

$L(y) < 1$: H_0 most likely

Likelihood function of signal in complex AWGN

Consider a received signal $y(t)$ with complex AWGN $n(t)$

- Check if it contains the signal $s(t)$

H$_0$ (dummy): Received signal is $y(t) = n(t)$

H$_1$: Received signal is $y(t) = s(t) + n(t)$

Theorem: The likelihood ratio with respect to the dummy hypothesis is

$$L(y|s) = e^{\frac{1}{\sigma^2}\left(Re\{<y,s>\} - \frac{\|s\|^2}{2}\right)}.$$

It is called the *likelihood function*.

Most Likely Hypothesis (1/2)

Finite number: M hypotheses

- H_i: The signal is $s_i(t)$
- Likelihood function $L(y|i)$
- Maximum likelihood (ML):

$$\hat{s}_{ML}(y) = s_k(t) \quad \text{where} \quad k = \text{argmax}_{1 \le i \le M} L(y|i)$$

Infinite number of hypotheses

This might be a vector

- Consider a signal $s(t,\Gamma)$ (for Γ from any set)
- Likelihood function $L(y|\Gamma)$
- Maximum likelihood (ML):

$$\hat{s}_{ML}(y) = s(t,\Gamma_*) \quad \text{where} \quad \Gamma_* = \text{argmax}_{\Gamma} \; L(y|\Gamma)$$

Most Likely Hypothesis (2/2)

Hypothesis testing with prior information

- H_i: The signal is $s_i(t)$
- H_i: Occurs with probability $\pi(i)$ ←

Finite hypotheses: Probability mass function
Infinite hypotheses: Probability density function

- Likelihood function $L(y|i)$
- Bayes' rule:

$$p(i|y) = \frac{L(y|i)\pi(i)}{p(y)}$$

Maximum a posteriori (MAP):

$$\hat{s}_{MAP}(y) = s(t, \Gamma_*) \quad \text{where} \quad \Gamma_* = \text{argmax}_\Gamma \; L(y|\Gamma)\pi(\Gamma)$$

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