TSKS04 Digital Communication
Continuation Course

Lecture 4

Channel Estimation and Complex Randomness

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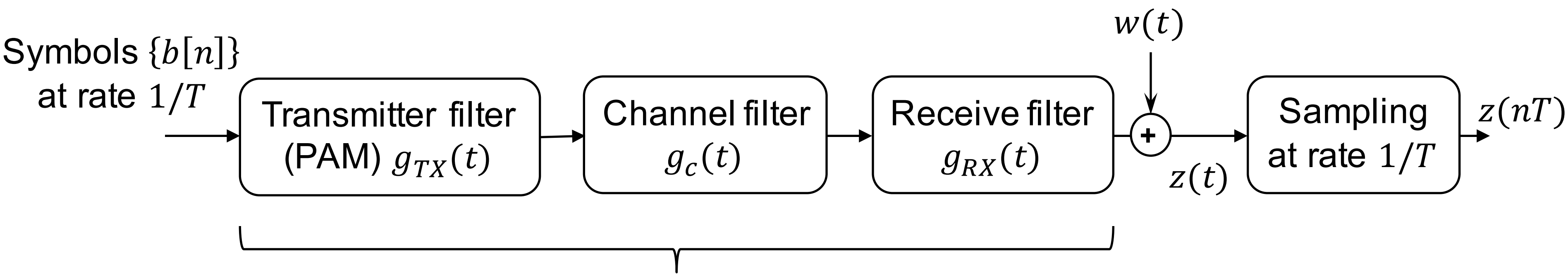
Recall: Channel model with ISI



Effective pulse shape:
$p(t) = (g_{TX} * g_c * g_{RX})(t)$

Signal model:

- $z(t) = \sum_m b[m]p(t - mT) + w(t)$

Sampled signal:

- $z(nT) = \sum_m b[m]\underbrace{p(nT - mT)} + w(nT)$

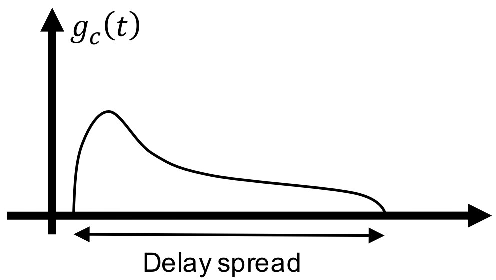
Describes the channel

How to estimate these parameters?

Estimating an L-tap Channel (1/2)

Channel has a delay spread

- $p(t)$ is non-zero when there are "echos"
- Assume that $p(jT) \neq 0$ for $j = 0, \ldots, L-1$



L-tap channel model (set $j = n - m$):

$$y[n] = \sum_{j=0}^{L-1} b[n-j]p(jT) + w(nT)$$

$$= \begin{pmatrix} b[n] & \ldots & b[n-L+1] \end{pmatrix} \underbrace{\begin{pmatrix} p(0) \\ \vdots \\ p((L-1)T) \end{pmatrix}}_{\boldsymbol{\theta} \,:\, L \text{ unknown parameters}} + w(nT)$$

Estimating an L-tap Channel (2/2)

- Make K observations: $y[0], y[1], \ldots, y[K-1]$
  - Gather all equations in matrix form:

$$\underbrace{\begin{pmatrix} y[0] \\ \vdots \\ y[K-1] \end{pmatrix}}_{=\mathbf{y}} = \underbrace{\begin{pmatrix} b[0] & \ldots & b[-L+1]) \\ \vdots & \ddots & \vdots \\ b[K-1] & \ldots & b[K-L]) \end{pmatrix}}_{=\mathbf{B}} \underbrace{\begin{pmatrix} p(0) \\ \vdots \\ p((L-1)T) \end{pmatrix}}_{=\boldsymbol{\theta}} + \underbrace{\begin{pmatrix} w(0) \\ \vdots \\ w((K-1)T) \end{pmatrix}}_{=\mathbf{w}}$$

K observations

Contains $L+K-1$ symbols: $b[-L+1], \ldots, b[K-1]$

L unknown parameters

K noise realizations: $\boldsymbol{w} \sim N(\mathbf{0}, \sigma^2 \boldsymbol{I})$

Observation: $\boldsymbol{y} = \boldsymbol{B}\boldsymbol{\theta} + \boldsymbol{w}$ → Similar to last time, except $\boldsymbol{\theta}$ is a vector

Estimation of Vector Parameters

Natural extension:

- Unknown parameter: $\boldsymbol{\theta} = (\theta[1], \dots \theta[L])^T$
- Observation: $\boldsymbol{y} = (y[1], \dots y[K])^T$ (realization of stochastic variable $\boldsymbol{Y}$)
- Relation: $p(\boldsymbol{y}|\boldsymbol{\theta})$ (conditional PDF)
- Estimator: $E\{\widehat{\boldsymbol{\theta}}(\boldsymbol{y})\}$

Unbiased estimator if:

$$E\{\widehat{\boldsymbol{\theta}}(\boldsymbol{y})\} = \boldsymbol{\theta}$$

Estimation error covariance matrix:

$$\boldsymbol{C} = E\left\{\left(\widehat{\boldsymbol{\theta}}(\boldsymbol{y}) - \boldsymbol{\theta}\right)\left(\widehat{\boldsymbol{\theta}}(\boldsymbol{y}) - \boldsymbol{\theta}\right)^T\right\}$$

CRLB for Vector Parameters

Theorem 2 (Cramer-Rao Lower Bound (CRLB)) *If the regularity condition* (B.36) *holds, then the estimation error covariance* $\mathbf{C}$ *of any unbiased estimator* $\hat{\boldsymbol{\theta}}(\mathbf{y})$ *must satisfy*

$$\mathbf{C} - \mathbf{I}^{-1}(\boldsymbol{\theta}) \succeq \mathbf{0} \tag{B.37}$$

where the notation $\succeq \mathbf{0}$ *means that the matrix is positive semidefinite.*[1] *The* (i,j)*th element of the* $L \times L$ *Fisher information matrix* $\mathbf{I}(\boldsymbol{\theta})$ *is given by*

$$[\mathbf{I}(\boldsymbol{\theta})]_{i,j} = -\mathrm{E}\left\{ \frac{\partial^2 \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j} \right\}. \tag{B.38}$$

An unbiased estimator that attains the CRLB can be found, for all $\boldsymbol{\theta}$*, if and only if the gradient of* $\ln p(\mathbf{y}|\boldsymbol{\theta})$ *can be decomposed as*

$$\frac{\partial \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{I}(\boldsymbol{\theta})\left(\mathbf{g}(\mathbf{y}) - \boldsymbol{\theta}\right) \tag{B.39}$$

for $\mathbf{I}(\boldsymbol{\theta})$ *defined in* (B.38) *and some* L*-dimensional function* $\mathbf{g}(\mathbf{y})$*. Then the MVU estimator is* $\hat{\boldsymbol{\theta}}(\mathbf{y}) = \mathbf{g}(\mathbf{y})$ *and has the estimation error covariance matrix* $\mathbf{I}^{-1}(\boldsymbol{\theta})$*.*

Solution to Channel Estimation Problem

Assume that $\mathbf{B}$ is known

- $b[-L+1], \ldots, b[K-1]$ are fixed training/pilot symbols
- Selected so $\mathbf{B}^T\mathbf{B}$ is invertible

Conditional PDF: $$p(\mathbf{y}|\boldsymbol{\theta}) = \frac{1}{(2\pi\sigma^2)^{K/2}} e^{-\frac{1}{2\sigma^2}(\mathbf{y}-\mathbf{B}\boldsymbol{\theta})^T(\mathbf{y}-\mathbf{B}\boldsymbol{\theta})}$$

- Gradient: $$\frac{\partial \ln p(\mathbf{y}|\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \frac{1}{\sigma^2}\mathbf{B}^T(\mathbf{y}-\mathbf{B}\boldsymbol{\theta})$$

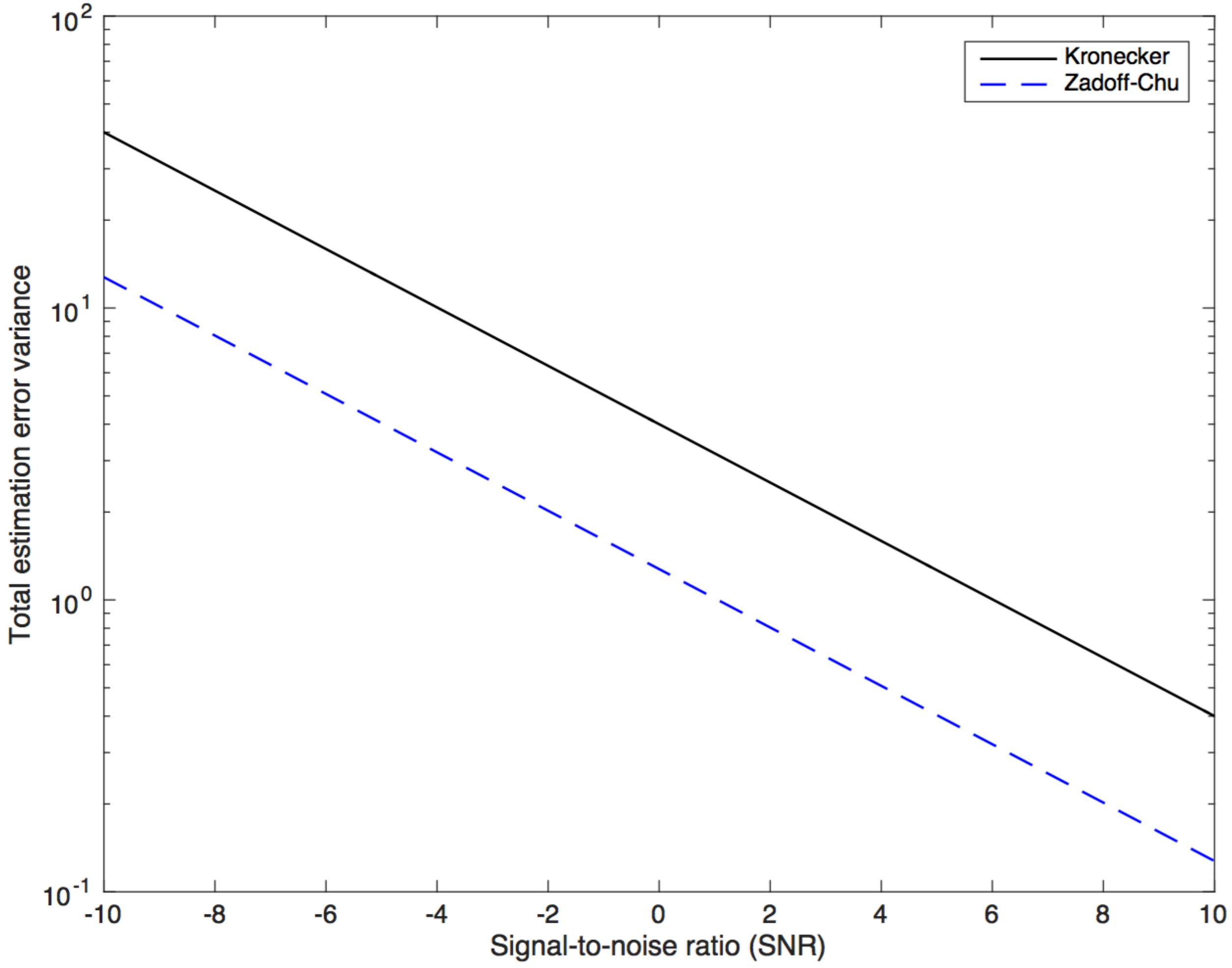
MVU estimator:

$$\hat{\boldsymbol{\theta}}(\mathbf{y}) = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{y}$$

Error covariance matrix:

$$\mathbf{C} = \sigma^2\left(\mathbf{B}^T\mathbf{B}\right)^{-1}$$

Example of Estimation Performance



Colored/Correlated Noise (1/2)

- What is colored noise?
  - Noise with a PSD that is not constant
  - White noise only exist in theory → Practical noise is colored
  - **Noise samples are correlated**

- Some sources
  - Shot noise, flicker noise, etc.
  - Also interference can be treated as noise

How to handle correlated noise?

Colored/Correlated Noise (2/2)

- How to model correlated additive noise?

$$\boldsymbol{y} = \boldsymbol{B}\boldsymbol{\theta} + \boldsymbol{w}$$

Known matrix ($\boldsymbol{B}$)

L unknown parameters ($\boldsymbol{\theta}$)

K noise realizations: $\boldsymbol{w} \sim N(\boldsymbol{0}, \boldsymbol{C}_w)$

- Some known full-rank covariance matrix $\boldsymbol{C}_w$

Pre-whitening

- Compute matrix square root $\boldsymbol{C}_w^{1/2}$
- Compute $\tilde{\boldsymbol{y}} = \left(\boldsymbol{C}_w^{1/2}\right)^{-1} \boldsymbol{y} = \left(\boldsymbol{C}_w^{1/2}\right)^{-1} \boldsymbol{B}\boldsymbol{\theta} + \left(\boldsymbol{C}_w^{1/2}\right)^{-1} \boldsymbol{w}$
- Note: $\left(\boldsymbol{C}_w^{1/2}\right)^{-1} \boldsymbol{w} \sim N(\boldsymbol{0}, \boldsymbol{I})$ and $\tilde{\boldsymbol{B}} = \left(\boldsymbol{C}_w^{1/2}\right)^{-1} \boldsymbol{B}$ is known

Estimation with Complex Numbers

- Estimation framework applicable when any type of numbers
  - Real numbers
  - Complex numbers
  - Binary numbers
  - …

How to handle complex AWGN?

Complex Gaussian Random Vectors

Definition: The $N \times 1$ complex random vector $\boldsymbol{x} = \boldsymbol{x}_c + j\boldsymbol{x}_s$ is *complex Gaussian* if the real vectors $\boldsymbol{x}_c, \boldsymbol{x}_s$ are jointly Gaussian.

Definition: A *proper* complex Gaussian random vector satisfies

$$E\{(\boldsymbol{x} - E\{\boldsymbol{x}\})(\boldsymbol{x} - E\{\boldsymbol{x}\})^T\} = 0$$

PDF with $\boldsymbol{m}_x = E\{\boldsymbol{x}\}$ and covariance $\boldsymbol{C}_x = E\{(\boldsymbol{x} - E\{\boldsymbol{x}\})(\boldsymbol{x} - E\{\boldsymbol{x}\})^H\}$:

$$p(\boldsymbol{x}) = \frac{1}{\pi^N \det(\boldsymbol{C}_x)} e^{-(\boldsymbol{x}-\boldsymbol{m}_x)^H \boldsymbol{C}_x^{-1} (\boldsymbol{x}-\boldsymbol{m}_x)}$$

Conjugate transpose

Notation: $\boldsymbol{x} \sim CN(\boldsymbol{m}_x, \boldsymbol{C}_x)$

Also known as:
Circularly-symmetric complex Gaussian

Proper Complex Gaussian Processes

Recall: $\boldsymbol{x} = (x_1, \dots, x_N)$ is *proper complex Gaussian*, denoted $\mathcal{CN}(\boldsymbol{m}_x, \boldsymbol{C}_x)$, if

$$p(\boldsymbol{x}) = \frac{1}{\pi^N \det(\boldsymbol{C}_x)} e^{-(\boldsymbol{x}-\boldsymbol{m}_x)^H \boldsymbol{C}_x^{-1} (\boldsymbol{x}-\boldsymbol{m}_x)}$$

Definition: A stochastic process is called *proper complex Gaussian* if $X(\bar{t}) = (X(t_1), \dots, X(t_N))$ is proper complex Gaussian for any $\bar{t} = (t_1, \dots, t_N)$

Theorem: A proper complex Gaussian that is wide sense stationary is also stationary in the strict sense.